THE OLYMPIAD CORNER

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Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by February 1, 2017.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC281. Find all polynomials $P(x)$ with real coefficients such that

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$.

OC282. Let $x, y, z$ be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum \left( \frac{x^2 - 1}{x} \right)^2 \geq 4.$$

OC283. In isosceles $\triangle ABC$, $AB = AC$, $I$ is its incenter, $D$ is a point inside $\triangle ABC$ such that $I, B, C, D$ are concyclic. The line through $C$ parallel to $BD$ meets $AD$ at $E$. Prove that $CD^2 = BD \cdot CE$.

OC284. A positive integer $n$ is given. If there exist sets $F_1, F_2, \ldots, F_m$ satisfying the following conditions, prove that $m \leq n$.

1. For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \ldots, n\}$
2. For all $1 \leq i < j \leq m$, $\min(|F_i - F_j|, |F_j - F_i|) = 1$

OC285. Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

OC281. Déterminer tous les polynômes $P(x)$ à coefficients réels tels que

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

pour tous réels $x$.

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OC282. Soit $x, y, z$ trois réels non nuls tels que $x + y + z = xyz$. Démontrer que
\[ \sum \left( \frac{x^2 - 1}{x} \right)^2 \geq 4. \]

OC283. Soit un triangle isocèle $ABC$ où $AB = AC$ et soit $I$ le centre du cercle inscrit dans le triangle. Soit $D$ un point à l’intérieur du triangle tel que $I, B, C$ et $D$ soient cocycliques. La droite qui passe au point $C$ et qui est parallèle à $BD$ coupe $AD$ en $E$. Démontrer que $CD^2 = BD \cdot CE$.

OC284. Soit $n$ un entier strictement positif. Sachant qu’il existe des ensembles $F_1, F_2, \cdots, F_m$ qui satisfont aux deux conditions suivantes, démontrer que $m \leq n$.

1. Pour tous $i$ ($1 \leq i \leq m$), on a $F_i \subseteq \{1, 2, \cdots, n\}$
2. Pour tous $i$ et $j$ ($1 \leq i < j \leq m$), on a $\min(|F_i - F_j|, |F_j - F_i|) = 1$

OC285. Étant donné un ensemble des carrés de 11 entiers, démontrer qu’il est possible de choisir six de ces carrés, $a^2, b^2, c^2, d^2, e^2$ et $f^2$, tels que
\[ a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}. \]
OLYMPIAD SOLUTIONS


OC221. From the point $P$ outside a circle $\omega$ with center $O$ draw the tangents $PA$ and $PB$ where $A$ and $B$ belong to $\omega$. In a random point $M$ in the chord $AB$ we draw the perpendicular to $OM$, which intersects $PA$ and $PB$ in $C$ and $D$. Prove that $M$ is the midpoint of $CD$.

Originally problem 3 of the 2014 Balkan Mathematical Olympiad Team Selection Test.

We received 10 correct submissions, consisting of a variety of solutions from many new readers which is fantastic!

We present the solution by Somasundaram Muralidharan, who actually gave 4 different solutions to this problem. The editor chose the shortest of the solutions to present.

Since $\angle OAC = \angle OMC = 90^\circ$, $O, A, C, M$ are concyclic. Similarly, since $\angle OMD = \angle OBD = 90^\circ$, the points $O, M, B, D$ are concyclic. Now,

\[ \angle MOC = \angle MAC = \angle MBP \]  
($PA, PB$ are tangents from an external point)

\[ = \angle MOD \]

Since $OM$ is perpendicular to $CD$, it follows that $M$ is the midpoint of $CD$.

OC222. Let \(a, b\) be natural numbers with \(ab > 2\). Suppose that the sum of their greatest common divisor and least common multiple is divisible by \(a + b\). Prove that the quotient is at most \(\frac{a+b}{4}\). When is this quotient exactly equal to \(\frac{a+b}{4}\)?

Originally problem 3 of the 2014 India National Olympiad.

We present the solution by Šefket Arslanagić. There were no other submissions.

First, if \(a = b\) then \(\text{lcm}(a,b) = \gcd(a,b) = a\) and thus the given condition is

\[
\frac{\text{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{a+a}{a+a} = 1 \leq \frac{a+b}{4} = \frac{a}{2}
\]

which holds whenever \(a \geq 2\) which must hold since \(ab > 2\). Equality holds here when \(a = b = 2\). Now, suppose without loss of generality that \(a < b\). Now, suppose that \(\gcd(a,b) = d\) and write \(a = a_1d\) and \(b = b_1d\) where \(a_1\) and \(b_1\) are coprime. Then \(\text{lcm}(a,b) = a_1b_1d\) and thus,

\[
\frac{\text{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{a_1b_1d + d}{a_1d + b_1d} = \frac{a_1b_1 + 1}{a_1 + b_1}
\]

If \(b_1 = a_1 + 1\), then the above becomes

\[
\frac{a_1b_1 + 1}{a_1 + b_1} = \frac{a_1^2 + a_1 + 1}{2a_1 + 1}
\]

which is a natural number. Hence, this value times 2 must also be a natural number. However,

\[
\frac{2a_1^2 + 2a_1 + 2}{2a_1 + 1} = a_1 + \frac{a_1 + 2}{2a_1 + 1}
\]

and thus, \(a_1 + 2 \geq 2a_1 + 1\). This implies that \(a_1 = 1\) and so \(a = d\) and \(b = 2d\) and \(d > 1\) since \(ab > 2\). Hence, in this case,

\[
\frac{\text{lcm}(a,b) + \gcd(a,b)}{a+b} = \frac{d + 2d}{d + 2d} = 1 \leq \frac{a+b}{4} = \frac{3d}{4}
\]

holding since \(d > 1\). Now, suppose that \(b_1 \geq a_1 + 2\). Then \(2 \leq b_1 - a_1\) and so
\(4 \leq b_1^2 - 2a_1b_1 + a_1^2\). Rearranging shows that \(a_1b_1 + 1 \leq \frac{(a_1+b_1)^2}{4}\) and hence

\[
\frac{a_1b_1 + 1}{a_1 + b_1} \leq \frac{a_1 + b_1}{4}.
\]

Hence, the given inequality holds. Equality holds in the cases

\[(a, b) \in \{(2, 2), (a_1d, (a_1+2)d), ((a_1+2)d, a_1d)\}\]

where \(d\) is a natural number and \(a_1\) is an arbitrary odd number (If it were even, then \(a_1\) and \(a_1 + 2\) are not coprime and so we could factor out another 2).
OC223. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Originally problem 3 from day 1 of the 2014 USAJMO.

We received 2 correct submissions. We present the solution by Oliver Geupel.

It is straightforward to check that the two functions $f : x \mapsto 0$ and $f : x \mapsto x^2$ are solutions. We prove that there are no other ones.

Suppose that $f$ is any solution. For integers $x$ and $y$, let $P(x, y)$ denote the assertion that $x$ and $y$ satisfy the proposed functional equation.

For every $x \neq 0$, the number $x$ divides $f(x)^2$ by $P(x, y)$. Assume towards a contradiction that $f(0) \neq 0$. Then from $P(2f(0), 0)$, we see that

$$2f(0)f(2f(0) - 2f(0)) + 0^2f(4f(0) - f(0)) = \frac{f(2f(0))^2}{2f(0)} + f(0f(0))$$

which simplifies to

$$4f(0) - 2 = \left( \frac{f(2f(0))}{f(0)} \right)^2.$$

This is a contradiction since the left hand side is divisible by exactly one copy of 2 whereas the right hand side must be divisible by 4. Hence $f(0) = 0$. From the assertions $P(x, 0)$ and $P(-x, 0)$ for $x \neq 0$, we obtain $f(-x) = \frac{f(x)^2}{x^2}$ and $f(x) = \frac{f(-x)^2}{x^2}$. Therefore, $f(x)^4 = x^6f(x)$, so that $f(x)$ is either 0 or $x^2$. Also, $f(x) = f(-x)$. Let us assume that $a$ and $b$ are non-zero integers such that $f(a) = 0$ and $f(b) = b^2$. All that remains to be done is to show that this is impossible.

By $P(x, a)$ for $x \neq 0$, we have $xf(-x) + a^2f(2x) = \frac{f(x)^2}{x^2}$. Thus, $f(2x) = 0$. Hence $b$ is odd. For every integer $x \neq 0$, we obtain $b^2f(4x - b^2) = f(b^3)$ applying $P(2x, b)$; whence $f(4x - b^2) = \frac{f(b^3)}{b^2}$. Since this holds for all nonzero $x$, we deduce that

$$f(4x - b^2) = 0.$$

If $b \equiv -1 \pmod{4}$ where $b \neq -1$, then putting $x = \frac{b^2 + b}{4}$ leads to

$$0 = f(4x - b^2) = f(b) = b^2 \neq 0,$$

where (1) is a contradiction since $f(b) = b^2$. Hence $b = 2$ and

$$f(4x - 2) = 0.$$

If $b = 2$, then putting $x = 1$ leads to

$$0 = f(2 - 1) = f(1) = 1^2 = 1.$$

Hence $f(x)$ is either 0 or $x^2$ for all $x \in \mathbb{Z}$.
a contradiction. If \( b \equiv 1 \pmod{4}, \) \( b \neq 1, \) then choosing \( x = \frac{b^2 + b}{4} \) gives (1), which is impossible. As a consequence, \( f(-1) = f(1) = 1 \) and \( f(x) = 0 \) for \( x \neq \pm 1. \) By \( P(2,1), \) we have \( 0 = 2f(0) + f(3) = f(1) = 1, \) which is the desired contradiction.

**OC224.** Let \( n > 1 \) be an integer. An \( n \times n \)-square is divided into \( n^2 \) unit squares. Of these unit squares, \( n \) are coloured green and \( n \) are coloured blue, and all remaining ones are coloured white. Are there more such colourings for which there is exactly one green square in each row and exactly one blue square in each column; or colourings for which there is exactly one green square and exactly one blue square in each row?

*Originally problem 5 of the 2014 South Africa National Olympiad.*

We received 2 correct submissions. We present the solution by Kathleen Lewis.

There are more colourings with one green and one blue in each row. To see this, think of first placing one green square in each row; for both methods there are \( n^n \) ways to do that. If we want to place a blue square in each row, there would be \((n-1)^n\) to accomplish this, since each row has one square already coloured green. But if we wish to put a blue square in each column, the number of possibilities depends on the arrangement already made of the green squares. Suppose that there are \( a_i \) blank squares in column \( i. \) Then the number of possible arrangements of the blue squares is \( \prod_{i=1}^n a_i. \) The total number of available squares is \( n^2 - n = n(n-1), \) so \( \sum_{i=1}^n a_i = n(n-1). \) But for variables with a fixed sum, the product is greatest when all the factors are equal. So, the maximum value of \( \prod_{i=1}^n a_i \) occurs when \( a_1 = a_2 = \cdots = a_n = n-1 \) and \( \prod_{i=1}^n a_i = (n-1)^n. \) In other cases, the product would be smaller, even as small as zero if the green squares were all placed in the same column. So the number of ways of placing a blue square in each column is always less than or equal to the number of ways to place the blue squares with one in each row.

**OC225.** Find the maximum value of real number \( k \) such that

\[
\frac{a}{1 + 9bc + k(b-c)^2} + \frac{b}{1 + 9ca + k(c-a)^2} + \frac{c}{1 + 9ab + k(a-b)^2} \geq \frac{1}{2}
\]

holds for all non-negative real numbers \( a, b, c \) satisfying \( a + b + c = 1. \)

*Originally problem 5 of the 2014 Japan Mathematical Olympiad.*

We received 3 correct submissions. We present the solution by Arkady Alt.

Let \( k \) be such that the original inequality holds for any non-negative real numbers \( a, b, c \) satisfying \( a + b + c = 1. \) Then, in particular, if \( a = 0 \) and \( b = c = 1/2, \) we get

\[
\frac{1/2}{1 + k(1/2)^2} + \frac{1/2}{1 + k(1/2)^2} \geq \frac{1}{2} \quad \iff \quad \frac{4}{k + 4} \geq \frac{1}{2} \quad \iff \quad k \leq 4.
\]
Let $k \leq 4$. By Cauchy’s Inequality

\[
\sum_{\text{cyc}} \frac{a}{1 + 9bc + k(b - c)^2} = \sum_{\text{cyc}} \frac{a^2}{a (1 + 9bc + k(b - c)^2)} \geq \frac{(a + b + c)^2}{\sum_{\text{cyc}} a (1 + 9bc + k(b - c)^2)} = \frac{1}{\sum_{\text{cyc}} a (1 + 9bc + k(b - c)^2)} = \frac{1}{1 + 9abc (3 - k) + k(ab + bc + ca)} = \frac{1}{1 + 9q (3 - k) + kp},
\]

where $p := ab + bc + ca$ and $q := abc$. We have

\[
p = ab + bc + ca \leq \frac{(a + b + c)^2}{3} = 1/3,
\]

\[
9q = 9abc \leq (ab + bc + ca) (a + b + c) = p,
\]

\[
9q \geq 4p - 1.
\]

(Schur’s Inequality $\sum_{\text{cyc}} a (a - b) (a - c) \geq 0$ in $p, q$ notation with normalization by $a + b + c = 1$).

If $k \leq 3$, then

\[
9q (3 - k) + kp \leq p (3 - k) + kp = 3p \leq 3 \cdot \frac{1}{3} = 1.
\]

If $3 < k \leq 4$, then

\[
9q (3 - k) + kp \leq (4p - 1) (3 - k) + kp = k + 3p (4 - k) - 3 \leq k + 3 \cdot \frac{1}{3} (4 - k) - 3 = 1.
\]

Thus,

\[
\sum_{\text{cyc}} \frac{a}{1 + 9bc + k(b - c)^2} \geq \frac{1}{1 + 9q (3 - k) + kp} \geq \frac{1}{1 + 1} = \frac{1}{2}
\]

for any $k \leq 4$ and, therefore, the maximum value of $k$ is 4.