Wobbling Bicycle: Solution

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**Problem.** A wobbling bicycle passes through a mud patch. One of its wheels traces a part of the curve $y = \sin x$. The other wheel makes a curve with a vertical inflection point. How long is the bicycle?

In order to eliminate effects due to bicycle geometry, tilting and wheel size, assume the bicycle has vanishingly thin tires with its front axle always positioned directly below a vertical headset. Assume also that both wheels were in the mud patch when the inflection point is traversed. Determine the distance $\ell$ between its axles.

**Solution.** One readily sees that the rear wheel traced the sine curve. Suppose the bicycle traveled from left to right on the coordinate plane, and that the front wheel is at $(X, Y)$ when the rear wheel is at $(x, \sin x)$, for $x \in [0, 2\pi]$. Then $(X, Y) = (x + \ell_x, \sin x + \ell_y)$, where $(\ell_x, \ell_y)$ is a translation vector of length $\ell$. Since the rear wheel of a bicycle is aligned with its frame, the line through $(x, \sin x)$ and $(X, Y)$ is tangent to the curve $y = \sin x$. This gives two equations,

$$\ell^2 = \ell_x^2 + \ell_y^2, \quad \frac{\ell_y}{\ell_x} = \frac{d}{dx} \sin x = \cos x.$$

Noting that $\ell_x > 0$, these imply $\ell_x = \ell/\sqrt{1 + \cos^2 x}$ and $X = x + \ell/(\sqrt{1 + \cos^2 x})$.

At the vertical inflection, the derivative function

$$\frac{dX}{dx} = 1 + \frac{\ell \cos x \sin x}{(1 + \cos^2 x)^{3/2}}$$

attains a local minimum value of 0. Since $1 + \cos^2 x$ is positive, the function

$$(1 + \cos^2 x)^3 - (\ell \cos x \sin x)^2$$

also attains a local minimum value of 0 at the vertical inflection. We change the variable to $C = \cos^2 x$ (so $\sin^2 x = 1 - C$). At the local minimum, $C$ is both a
zero and a critical point of the resulting function $f(C)$.

\[
\begin{align*}
  f(C) &= (1 + C)^3 - \ell^2 C(1 - C) = 0 \quad (2) \\
  f'(C) &= 3(1 + C)^2 - \ell^2 (1 - 2C) = 0. \quad (3)
\end{align*}
\]

To solve for $\ell^2$, we subtract $1 + C$ times equation (3) from 3 times equation (2) to get

\[
\ell^2 (1 - 4C + C^2) = 0.
\]

Since $\ell > 0$, the second factor equals zero. We “complete the square” in two ways,

\[
\begin{align*}
  (1 + C)^2 &= (1 - 4C + C^2) + 6C = 6C, \\
  (1 - 2C)^2 &= (1 - 4C + C^2) + 3C^2 = 3C^2.
\end{align*}
\]

Using these with (3), we find

\[
\ell^2 = \frac{3(1 + C)^2}{1 - 2C} = \frac{3 \cdot 6C}{\pm \sqrt{3}C^2} = \pm 6\sqrt{3} = \pm \sqrt{108}.
\]

Thus the bicycle’s “length” is $\ell = \sqrt{108} \approx 3.223709$.

\[\square\]

**Note:** With a bit more work, one finds that

\[
C = 2 - \sqrt{3} = \frac{1}{4} (\sqrt{6} - \sqrt{2})^2 \approx 0.268,
\]

and that at the vertical inflection we have the following values.

\[
\begin{align*}
  x &= \arccos \left(-\sqrt{2 - \sqrt{3}}\right) = \arccos \frac{\sqrt{2} - \sqrt{6}}{2} = \pi - \arcsin \sqrt{3} - 1 \approx 2.115, \\
  y &= \sqrt{3} - 1 \approx 0.856, \\
  l_x &= \sqrt{3\sqrt{3} + 3} \approx 2.863, \\
  l_y &= -\sqrt{3\sqrt{3} - 3} \approx -1.482, \\
  X &= \arccos \frac{\sqrt{2} - \sqrt{6}}{2} + \sqrt{3\sqrt{3} + 3} \approx 4.978, \\
  Y &= \sqrt{\sqrt{3} - 1} - \sqrt{3\sqrt{3} - 3} = -\left(\sqrt{3} - 1\right)^{3/2} \approx -0.626.
\end{align*}
\]

It is notable that, at the vertical inflection, the vertical positions of the tires satisfy

\[
Y + y^3 = 0.
\]