Constructing Paradoxical Sequences

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Mathematical olympiads sometimes include seemingly paradoxical problems. Consider the following example:

Problem 1. For the last year, Shane tracked his monthly income and his monthly expenses. Is it possible that for all sets of five consecutive months his expenses exceeded his income, but his yearly income exceeded his overall expenses?

At first sight, this seems impossible. However, after some consideration, we can construct an example that satisfies the conditions of the problem. Suppose that the monthly balance (the difference between the income and the expenses) takes on only two values $x > 0$ and $y < 0$. Let us construct a sequence of length 12 using x and y, which contains only one y for any 5 consecutive spots. For example,

$$x, x, x, x, y, x, x, x, x, y, x, x.$$

Next, we need to make sure that the sum of any 5 consecutive numbers is negative, that is $y + 4x < 0$, and that the sum of all 12 numbers is positive, so $2y + 10x > 0$. Overall, we get

$$-5x < y < -4x.$$

So we can take, say, $x = 2$ and $y = -9$.

This problem used the sequence of length 12, but how long can a sequence be with such conditions imposed on it? Let us take a look at a more general problem.

Problem 2. (Inspired by one of the problems from the XIX International Mathematical Olympiad). Consider a sequence of real numbers in which the sum of any seven consecutive elements is negative, but the sum of any eleven consecutive elements is positive.

a) What is the maximum possible length of such a sequence?

b) Give an example of such a sequence of length 16.

You can approach part b) of Problem 2 using trial and error method, but with some creativity. However, you can also approach it quite systematically. For example, let us find a sequence $x_1, x_2, \ldots, x_{16}$ that satisfies the conditions

$$\begin{aligned} &x_1 + x_2 + \cdots + x_7 = -b, \\ &\cdots \\ &x_{10} + x_{11} + \cdots + x_{16} = -b, \\ &x_1 + x_2 + \cdots + x_{11} = a, \\ &\cdots \\ &x_6 + x_7 + \cdots + x_{16} = a, \end{aligned}$$

where a and b are some positive constants.

Exercise 1. Solve this system of equations.

The reader who struggles with this system will appreciate how difficult it is to solve it. Below we will show another method that will help us easily find an example for Problem 2b. But first we will take a closer look at Problem 2a.

Consider a more general problem and define some terms. Given a sequence of m numbers, we say that a *q-sum* is a sum of some q consecutive numbers in this sequence. A sequence where any n-sum has one sign and any k-sum has the opposite sign is called an *$\{n, k\}$-sequence*.

Problem 3. Prove that the length m of an $\{n, k\}$-sequence does not exceed $n + k - d - 1$, where d is the greatest common divisor of n and k.

First of all, note that that neither of n nor k can be divisible by the other (prove this). We will start off by proving that there cannot be an $\{n, k\}$-sequence of length m if $m > n + k - d - 1$.

Suppose that we found a sequence of length $n + k - d$ that satisfies the necessary conditions. Take the smaller of the two numbers n and k, suppose it is k. Remove the first k numbers from our sequence. In the remaining sequence of $n-d$ numbers, all k-sums still have the same sign as before and all $(n-k)$-sums have the opposite signs (prove this statement by contradiction). Since $n - d = (n - k) + k - d$, we transformed the $\{n, k\}$-sequence into a $\{n-k, k\}$-sequence. Repeating the process, we get a chain of sequences of decreasing lengths $\{n, k\} \to \{n_1, k_1\} \to \{n_2, k_2\} \to \cdots \to \{n_l, k_l\} \to$, where one of the numbers n_l or k_l in the end equals to d and the other one is divisible by d. But this is a contradiction.

Obviously, if there is no sequence of length $n + k - d$ satisfying the conditions, there is no longer sequence satisfying these conditions.

We can give an example of such sequence of length $n + k - d - 1$. We look for this example among the sequences whose elements can take only one of two values x and y, which we will pick later. As our "guiding star", we take the chain $\{n, k\} \to \{n_1, k_1\} \to \{n_2, k_2\} \to \cdots \to \{n_l, k_l\} \to$ from above; we will call it *the defining sequence*. This chain realizes the familiar Euclidean algorithm computing the greatest common divisor d of two numbers n and k. Indeed, $d = |n_l - k_l|$. Here, we will use this chain for constructing our sequence and we will do so by moving from the end of the chain to its beginning. First, using the pair (n_l, k_l), let us construct a sequence

$$x, x, \ldots, x, y, x, x, \ldots, x, \tag{1}$$

the number of x's on the left of y is the same as on the right and equals

$$\begin{cases} n_l - 1, & \text{if } k_l > n_l, \\ k_l - 1, & \text{if } n_l > k_l. \end{cases}$$

In general, we say a sequence is *$[p, q]$-sequence* if all of its subsequences of length p contains the same number of y's and all of its subsequences of length q contains the same number of y's (the number of y's need not be the same in both cases). Clearly, the sequence (1) is $[n_l, k_l]$-sequence: any subsequence of length n_l or k_l contains exactly one y. This definition characterizes the uniformity of distributions of y's in the sequence.

Starting with the base sequence (1), we will add to it according to the following rule. Suppose we already have a $[n_i, k_i]$-sequence of x's and y's, which corresponds to the pair of numbers (n_i, k_i) from the defining sequence. We can get to this sequence from the previous pair (n_{i-1}, k_{i-1}) in two ways: either $n_i = n_{i-1} - k_{i-1}$ or $k_i = k_{i-1} - n_{i-1}$. Let us consider the first case (the second one is analogous). To increase the length of the sequence, fix the first k_{i-1} symbols and append them to the sequence from the left. We will prove that this results in a $[n_i, k_i]$-sequence.

Any subsequence of the resulting sequence consisting of $k_{i-1} = k_i$ consecutive symbols contains the same number of y's. Any subsequence of the resulting sequence consisting of $n_{i-1} = n_i + k_{i-1}$ consecutive symbols can be thought of as composed of two parts: right subsequence of length n_i and left subsequence of length k_{i-1}. Each of these subsequences contains a fixed number of y's, so the entire sequence contains a fixed number of y's.

We now come back to solve Problem 2b). Since $(7, 11) \to (7, 4) \to (3, 4)$, the first step of the algorithm gives

$$x, x, y, x, x.$$

The second step gives

$$x, x, y, x, x, x, y, x, x,$$

and the third step produces

$$x, x, y, x, x, x, y, x, x, y, x, x, x, y, x, x.$$

Since the sum of any seven consecutive elements must be negative and the sum of any eleven consecutive elements must be positive, the following inequalities must be satisfied:

$$\begin{cases} 2y + 5x < 0, \\ 3y + 8x > 0, \end{cases}$$

which gives $-\frac{5}{2}x > y > -\frac{8}{3}x$. We can take, for example, $x = 5$ and $y = -13$.

To prove the validity of the above algorithm, it only remains to prove that the general system of inequalities for x and y always has solutions. We will leave this proof as an exercise for the reader.

Exercise 2. Suppose the aforementioned algorithm gives a sequence of x's and y's. Let a denote the number of y's in any subsequence of length n and let b denote the number of y's in any subsequence of length k. Prove that the following two systems of inequalities always have solutions:

$$\begin{cases} ay + (n-a)x > 0, \\ by + (k-b)x < 0, \end{cases} \quad \text{and} \quad \begin{cases} ay + (n-a)x < 0, \\ by + (k-b)x > 0. \end{cases}$$

Exercise 3. It is known that for some sequence of length 23, the sum of any ten consecutive elements is negative and the sum of any k consecutive elements is positive. Find k, if the length of this sequence is the largest possible. Give an example of such a sequence.

Exercise 4. It is known that for some sequence the sum of any n consecutive elements is negative and the sum of any k consecutive elements is positive. It is also known that the maximum possible length of this sequence is 30. Find the maximum possible value of the difference $|k - n|$. Give an example of such a sequence.

Problem 4. Consider an integrable function $f(x)$ on the interval $[0, 23]$. Suppose that the definite integral of $f(x)$ on any interval of length n is positive, while the definite integral of $f(x)$ on any interval of length m is negative, where $c > m > n > 0$. Suppose the fraction $\frac{m}{n}$ reduces to $\frac{q}{p}$ in lowest terms; that is, p and q are mutually prime positive integers. Prove that $c < m + n - \frac{m}{q}$.

Let $d = \frac{m}{q} = \frac{n}{p}$, so that $n = pd$ and $m = qd$. Suppose that $c \geq m + n - \frac{m}{q} = (p + q - 1)d$. For any positive integer k such that $kd \leq c$, let

$$S_k = \int_{(k-1)d}^{kd} f(x)dx$$

and consider the following sequence of length $p + q - 1$: $S_1, S_2, \ldots, S_{p+q-1}$. It is not hard to check that this sequence satisfies the properties that the sum of any p consecutive elements is positive and the sum of any q consecutive elements is negative. By using the result from Problem 3, we see that such a sequence has maximum possible length of $p + q - 2$. This contradiction finishes the proof.

Exercise 5. Consider an integrable function $f(x)$ on the interval $[0, 23]$. Suppose that the definite integral of $f(x)$ on any interval of length 10 is positive, while the definite integral of $f(x)$ on any interval of length 16 is negative. Is this possible? What if $x \in [0, 24]$?

Exercise 6. Does there exist a continuous function $y = f(x)$ such that any definite integral of $f(x)$ on any interval of length 3 is negative, while the definite integral of $f(x)$ on any interval of length 5 is positive?

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