OC252. Pour un triangle obtus $ABC$ ($AB > AC$), soit $O$ le centre du cercle circonscrit et soit $D, E$ et $F$ les mi points de $BC$, $CA$ et $AB$ respectivement. La médiane $AD$ intersecte $OF$ et $OE$ à $M$ et $N$ respectivement; $BM$ rencontre $CN$ au point $P$. Démontrer que $OP \perp AP$.

OC253. Démontrer qu'il existe un nombre infini d'entiers positifs $n$ tels que $3^n + 2$ et $5^n + 2$ sont composés.

OC254. Déterminer tous les entiers non négatifs $k$ et $n$ satisenant $2^{2k+1} + 9 \cdot 2^k + 5 = n^2$.

OC255. Soit $n$ un entier positif et soit $x_1, x_2, \ldots, x_n$ des nombres réels positifs. Démontrer qu'il existe des nombres $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ tels que l'inégalité suivante tient:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \geq (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2.$$
Let $P = \alpha A + \beta B + \gamma C$ where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Since the point $P - \alpha A = \beta B + \gamma C$ is on $AP$ and on $BC$, we have

$$(1 - \alpha)D = P - \alpha A = \beta B + \gamma C$$

and so $\frac{AP}{PD} = \frac{1 - \alpha}{\alpha}$. Similarly,

$$(1 - \gamma)F = P - \gamma C = \alpha A + \beta B$$

and so $\frac{AF}{FB} = \frac{\beta}{1 - \gamma}$.

Since the altitudes $FH$ and $BK$ in triangles $AFP$ and $BPD$, respectively, satisfy $\frac{FH}{BK} = \frac{AF}{AB}$ (note that $\triangle AFH$ and $\triangle ABK$ are similar), we obtain

$$1 = \frac{[APF]}{[BPD]} = \frac{FH \cdot AP}{BK \cdot PD} = \frac{AF \cdot AP}{AB \cdot PD} = \frac{\beta(1 - \alpha)}{\alpha(1 - \gamma)}$$

where $[XYZ]$ denotes the area of $\triangle XYZ$. It follows that $\alpha(\alpha + \beta) = \beta(\beta + \gamma)$; in the same way, we have $\beta(\beta + \gamma) = \gamma(\gamma + \alpha)$.

Now, let $k = \alpha^2 + \alpha \beta = \beta^2 + \beta \gamma = \gamma^2 + \gamma \alpha$. Then, recalling that $\alpha + \beta + \gamma = 1$, we successively obtain

$$k = k\beta + k\gamma + k\alpha = \alpha^2 \beta + \alpha \beta^2 + \beta^2 \gamma + \beta \gamma^2 + \gamma^2 \alpha + \gamma \alpha^2$$

and

$$k = k\gamma + k\alpha + k\beta = \alpha^2 \gamma + \alpha \beta \gamma + \beta^2 \alpha + \alpha \beta \gamma + \gamma^2 \beta + \alpha \beta \gamma.$$

As a result, we have

$$\alpha^2 \beta + \beta^2 \gamma + \gamma^2 \alpha = 3\alpha \beta \gamma = 3\sqrt[3]{\alpha^2 \beta \cdot \beta^2 \gamma \cdot \gamma^2 \alpha}.$$

From the case of equality in AM-GM, this means that $\alpha^2 \beta = \beta^2 \gamma = \gamma^2 \alpha$. This easily yields $\alpha^3 = \alpha \beta \gamma = \beta^3 = \gamma^3$ and so $\alpha = \beta = \gamma$. Thus, $P$ is the centroid of $\triangle ABC$.

**OC192.** Find all possible values of a positive integer $n$ for which the expression $S_n = x^n + y^n + z^n$ is constant for all real $x, y, z$ with $xyz = 1$ and $x + y + z = 0$.

*Originally problem 2 from the 2013 Spain Mathematical Olympiad.*

We received two correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We claim that the only solutions are $n = 1$ and $n = 3$ (and $n = 0$ if non-negative integers are allowed). Let $\sigma_1 = x + y + z$, $\sigma_2 = xy + xz + yz$, and $\sigma_3 = xyz$ (the elementary symmetric functions in $x, y, z$). It is straightforward to verify that for any $x, y, z$,

$$S_n = \sigma_1 S_{n-1} - \sigma_2 S_{n-2} + \sigma_3 S_{n-3}.$$

In our case, this gives

$$S_n = -\sigma_2 S_{n-2} + S_{n-3}. \quad (1)$$

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We have $S_0 = 3$, $S_1 = \sigma_1 = 0$, and $S_2 = \sigma^2 - 2\sigma = -2\sigma$. By recurrence (1), we have $S_1 = -\sigma_2 S_1 + S_0 = 0 + 3 = 3$.

We claim that for $n > 3$, $S_n$ is never constant. We first note that $\sigma_2$ is not constant. The values $x = -1$, $y = (1 + \sqrt{5})/2$, $z = (1 - \sqrt{5})/2$ satisfy the conditions of the problem and here $\sigma_2 = -2$. On the other hand, $x = 2, y = (-2 + \sqrt{2})/2, z = (-2 - \sqrt{2})/2$ also satisfy the conditions, but $\sigma_2 = -7/2$. Since $\sigma_2$ is not constant, the following lemma will suffice to prove our claim.

**Lemma.** For $k \geq 1$, $S_{2k}$ is a polynomial of degree $k$ in $\sigma_2$ with leading coefficient $(-1)^k \cdot 2$ and $S_{2k+1}$ is a polynomial of degree $k - 1$ in $\sigma_2$ with leading coefficient $(-1)^{k-1}(2k + 1)$.

**Proof.** We have $S_2 = -2\sigma_2$ and $S_3 = 3$, so the result follows when $k = 1$. Assuming the result holds for all $k < N$, we have

$S_{2N} = -\sigma_2 S_{2N-2} + S_{2N-3}$.

Since by the induction hypothesis, $S_{2N-2}$ is polynomial in $\sigma_2$ of degree $N - 1$ with leading coefficient $(-1)^{N-1} \cdot 2$ and $S_{2N-3}$ is a polynomial in $\sigma_2$ of degree $N-2$, $S_{2N}$ is a polynomial of degree $N$ with leading coefficient $(-1)(-1)^{N-1} \cdot 2 = (-1)^N \cdot 2$. Similarly,

$S_{2N+1} = -\sigma_2 S_{2N-1} + S_{2N-2}$

with $-\sigma_2 S_{2N-1}$ being a polynomial of degree $N - 1$ with leading coefficient

$-(-1)^{N-2}(2N - 1) = (-1)^{N-1}(2N - 1)$

and $S_{2N-2}$ is a polynomial of degree $N - 1$ with leading coefficient $(-1)^{N-1} \cdot 2$, so their sum is a polynomial of degree $N - 1$ with leading coefficient

$(-1)^{N-1}(2N - 1) + (-1)^{N-1} \cdot 2 = (-1)^{N-1}(2N + 1)$.

**OC193.** Let $\{a_n\}$ be a positive integer sequence such that $a_{i+2} = a_{i+1} + a_i$ for all $i \geq 1$. For positive integer $n$, define $\{b_n\}$ as

$b_n = \frac{1}{a_{2n+1}} \sum_{i=1}^{4n-2} a_i$.

Prove that $b_n$ is a positive integer, and find the general form of $b_n$.

Originally problem 4 from day 1 of the 2013 Korea National Olympiad.

We present the solution by Ángel Plaza. There were no other submissions.

From the definition of $\{a_n\}$ it is deduced that $a_n = F_{n-2}a_1 + F_{n-1}a_2$ for $n > 2$ where $F_n$ is the $n$-th Fibonacci number beginning with $F_1 = F_2 = 1$. This fact can be proved easily by induction.

*Crux Mathematicorum*, Vol. 41(9), November 2015
Then, using well known facts about Fibonacci numbers,

\[
4n - 2 \sum_{i=1}^{4n-2} a_i = \left(1 + \sum_{i=1}^{4n-4} F_i\right) a_1 + \left(\sum_{i=1}^{4n-3} F_i\right) a_2 = F_{4n-2}a_1 + (F_{4n-1} - 1)a_2.
\]

On the other hand, \(a_{2n+1} = F_{2n}a_1 + F_{2n}a_2\). Finally, since \(L_{2n-1}F_{2n} = F_{4n-1} - 1\), it follows that \(b_n = L_{2n-1}\), where \(L_n\) is the \(n\)-th Lucas number beginning with \(L_1 = 2\) and \(L_2 = 1\) and the problem is done.

**OC194.** Let \(Q^+\) be the set of all positive rational numbers. Let \(f : Q^+ \to \mathbb{R}\) be a function satisfying the following three conditions:

1. for all \(x, y \in Q^+\), \(f(x)f(y) \geq f(xy)\);
2. for all \(x, y \in Q^+\), \(f(x+y) \geq f(x) + f(y)\);
3. there exists a rational number \(a > 1\) such that \(f(a) = a\).

Prove that \(f(x) = x\) for all \(x \in Q^+\).

*Originally problem 5 from day 2 of the 2013 International Mathematical Olympiad.*

*There were no submitted solutions.*

**OC195.** Let \(O\) denote the circumcentre of an acute-angled triangle \(ABC\). Let point \(P\) on side \(AB\) be such that \(\angle BOP = \angle ABC\), and let point \(Q\) on side \(AC\) be such that \(\angle COQ = \angle ACB\). Prove that the reflection of \(BC\) in the line \(PQ\) is tangent to the circumcircle of triangle \(APQ\).

*Originally problem 5 from the 2013 Canadian Mathematical Olympiad.*

*We received two correct submissions. We present the solution by Michel Bataille.*

Let \(\Gamma\) and \(\Gamma'\) be the circumcircles of \(\Delta ABC\) and \(\Delta APQ\), respectively. Let \(O'\) be the centre of \(\Gamma'\) and let \(a = \angle BAC\), \(\beta = \angle CBA\) and \(\gamma = \angle BCA\). Since \(\Delta ABC\) is acute-angled, \(\angle BOC = 2\alpha\) and so

\[\angle POQ = 360^\circ - 2\alpha - \beta - \gamma = 180^\circ - \alpha.\]

It follows that \(O\) lies on \(\Gamma'\).

Let \(A'\) be the second point of intersection of \(\Gamma\) and \(\Gamma'\). From Focus On... No 12, the spiral similarity with centre \(A'\) transforming \(\Gamma\) into \(\Gamma'\) transforms \(B\) into \(P\) and \(C\) into \(Q\) and so \(\angle (BC, PQ) = \angle (A'O, A'O')\). (Here and in what follows, \(\angle (\ell, \ell')\) denotes the directed angle of lines from the line \(\ell\) to the line \(\ell'\) and we suppose that the orientation is such that \(\angle (AB, AC) = \alpha, \angle (BC, BA) = \beta, \angle (CA, CB) = \gamma).
Assume that $A'O \perp PQ$ has been proved. Then, since the tangent $t$ to $\Gamma'$ at $A'$ is perpendicular to $A'O'$, we have $\angle(A'O, A'O') = \angle(PQ, t)$. It follows that $\angle(BC, PQ) = \angle(PQ, t)$, hence the reflection of $BC$ in $PQ$ is $t$. Therefore, it is sufficient to prove the assumption $A'O \perp PQ$.

We observe that $OB = OA$ and $\angle(BC, BO) = 90^\circ - \alpha$, hence

$$\angle(BO, BA) = \angle(AB, AO) = \angle(AP, AO) = \beta - (90^\circ - \alpha) = 90^\circ - \gamma.$$  

Similarly,

$$\angle(CA, CO) = \angle(CQ, CO) = \angle(AO, AQ) = 90^\circ - \beta.$$  

Since $O, A, A'$ and $Q$ are concyclic, we obtain

$$\angle(A'O, A'Q) = \angle(AO, AQ) = \angle(CQ, CO).$$  

Thus, the circle $\Gamma'$ is the locus of all points $M$ such that $\angle(MO, MQ) = 90^\circ - \beta$ and since $\angle(CO, CQ) = -\angle(AO, AQ)$, $C$ is on the reflection of $\Gamma'$ in the line $OQ$. Since $\Gamma$ is its own reflection in its diameter $OQ$, the reflection of $C$ in $OQ$ is on $\Gamma$ and $\Gamma'$, hence is $A'$ (not $A$, unless $CQ \perp OQ$ but then $\gamma = \angle COQ = 90^\circ - \angle OCQ = \beta$, hence $AB = AC$, in which case it is easily seen that $A = A'$). As a result, $\angle(OQ, OA') = \angle(OQ, OQ) = \gamma$ and so

$$\angle(PQ, A'O) = \angle(QP, QO) + \angle(OQ, OA')$$
$$= \angle(AP, AO) + \angle(OQ, OQ)$$
$$= 90^\circ - \gamma + \gamma = 90^\circ.$$  

and we are done.