THE CONTEST CORNER

No. 39

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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by November 1, 2016, although late solutions will also be considered until a solution is published.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC191. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

CC192. Let M be a $3 \times 3$ matrix with all entries drawn randomly (and with equal probability) from \{0, 1\}. What is the probability that $\det M$ will be odd?

CC193. Consider the set of numbers \{1, 2, \ldots, 10\}. Let \{a_1, a_2, \ldots, a_{10}\} be some permutation of these numbers and compute

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$ 

What is the maximum possible value of the above sum over all possible permutations and how many permutations give you this maximum value?

CC194. At a strange party, each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the party that they both knew. For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew. How many people were at the party?

CC195. A bisecting curve is one that divides a given region into two subregions of equal area. The shortest bisecting curve of a circle is clearly a diameter. What is the shortest bisecting curve of an equilateral triangle?

CC191. Il y a 32 concurrents dans un tournoi et il n’y a pas deux concurrents de force égale. Dans n’importe quel match entre deux concurrents, le plus fort l’emporte toujours. Démontrer qu’il est possible de déterminer les récipiendaires des médailles d’or, d’argent et de bronze après 39 parties.
CC192. Soit $M$ une matrice $3 \times 3$ dont les éléments sont tous choisis de façon aléatoire dans l’ensemble $\{0, 1\}$. Quelle est la probabilité pour que $\det M$ soit impair?

CC193. On considère l’ensemble $\{1, \ldots, 10\}$ et une permutation $\{a_1, \ldots, a_{10}\}$ de cet ensemble. On calcule

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

Parmi toutes les permutations de l’ensemble, quelle est la valeur maximale de cette somme et combien de permutations donnent cette valeur maximale?

CC194. Lors d’une drôle de fête, chaque personne connaît 22 autres personnes. Pour chaque paire de personnes $X$ et $Y$ qui se connaissent l’une l’autre, il n’y a aucune autre personne à la fête que $X$ et $Y$ connaissent tous les deux. Pour chaque paire de personnes $X$ et $Y$ qui ne se connaissent pas l’une l’autre, il y a 6 personnes à la fête que $X$ et $Y$ connaissent tous les deux. Combien y a-t-il de personnes à la fête?

CC195. Une courbe bissectrice d’une surface est une courbe qui coupe la surface en deux régions de même aire. La courbe bissectrice d’un cercle est évidemment un diamètre. Quelle est la courbe bissectrice la plus courte d’un triangle équilatéral?

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CONTEST CORNER

SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(9), p. 368–369. All problems are from 42nd Ural tournament of young mathematicians, as printed in Kvant 2014(1).

CC141. Alice writes down 100 consecutive natural numbers. Bob multiplies 50 of them: 25 smallest ones and 25 largest ones. He then multiplies the remaining 50 numbers. Can the sum of the two products be equal to $100! = 1 \cdot 2 \cdot \ldots \cdot 100$?

Problem 2 of grade 7 level of 42nd Ural tournament of young mathematicians.

We received three solutions, two of which were correct. We present a composite of the two correct solutions from Konstantine Zelator and Fernando Ballesta Yagüe.

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Write the 100 consecutive natural numbers as $n + 1, n + 2, \ldots, n + 100$, where $n$ is a natural number. Let $P_1 = (n + 1)(n + 2) \cdots (n + 25)$ be the product of the 25 smallest numbers, $P_2 = (n + 26)(n + 27) \cdots (n + 75)$ be the product of the 50 middle numbers, and $P_3 = (n + 76)(n + 77) \cdots (n + 100)$ be the product of the largest 25 numbers. We want to know if it’s possible to have $P_1 \cdot P_3 + P_2 = 100!$.

The factors in the products $P_1$, $P_2$ and $P_3$ contain 100 consecutive natural numbers. Thus, if $P_1 \cdot P_3 + P_2 = 100!$, the greatest prime number smaller than 100, which is 97, must be a divisor of each of $P_1 \cdot P_3$ and $P_2$.

If 97 is a divisor of $P_2$, that means it is a divisor of one of its factors $n + k$, $k \in 26, \ldots, 75$. The neighbouring multiples of 97 are $n + k - 97$ and $n + k + 97$. Both of these fall outside the range of the natural numbers in $P_1$ (which, for all options of $n + k$ in $P_2$, must fall within $n + k - 74$ to $n + k - 1$) and $P_3$ (which must fall within $n + k + 1$ to $n + k + 74$). Thus if 97 is a divisor of $P_2$, then it cannot be a divisor of either $P_1$ or $P_3$. We cannot have $P_1 \cdot P_3 + P_2 = 100!$.

CC142. Roboto writes down a number. Every minute, he increases the existing number by the double of the number of its natural divisors (including 1 and itself). For example, if he started with 5, the sequence would be 5, 9, 15, 23, \ldots. What is the maximum number of perfect squares that appears on the board within 24 hours?

*Problem 3 of grade 7 level of 42nd Ural tournament of young mathematicians.*

We received one correct solution. We feature the solution by Konstantine Zelator.

We show that such a sequence contains at most one perfect square. We make two observations. First, if $n$ is a perfect square, then $n \equiv 0$ or 1 (mod 4). Second, the number of positive divisors of an integer is odd if and only if that integer is a perfect square.

Suppose such a sequence contains a perfect square $k$. Then $k$ has $2m + 1$ divisors for some integer $m$, and twice the number of divisors of $k$ is $4m + 2$. Since $k$ is a perfect square $k \equiv 0$ or 1 (mod 4) and $k + 4m + 2 \equiv 2$ or 3 (mod 4). We notice that numbers of these forms cannot be perfect squares, and so have an even number of divisors. Thus, twice the number of their divisors is a multiple of 4. So after this point, the terms in the sequence will be the same modulo 4 and hence can never be perfect squares. Thus, there can be at most one perfect square in the sequence.

CC143. Summer Camp has attracted 300 students this year. On the first day, the students discovered (as mathematicians would) that the number of triples of students who mutually know each other is greater than the number of pairs of students who know each other. Prove that there is a student who knows at least 5 other students.

*Problem 9 of grade 7 level of 42nd Ural tournament of young mathematicians.*

We received no solutions to this problem.
CC144. Year 2013 is the first one since Middle Ages that uses 4 consecutive digits in its base 10 representation. How many other years like this will there be before year 10,000?

Problem 1 of grade 8 level of 42nd Ural tournament of young mathematicians.

We received three solutions, of which two were correct and complete. We present the most succinct, by Andrea Fanchini.

We have seven possible groups of four consecutive digits:

\[
\{0123, 1234, 2345, 3456, 4567, 5678, 6789\}.
\]

For each group we have \(4! = 24\) possible permutations. In total, this makes \(7 \cdot 24 = 168\) possibilities. But from these we have to cancel those that start with 0, those that start with 1, and the year 2013.

The possibilities that start with 0 are \(3! = 6\) permutations (0 is present only in the first group). The possibilities that start with 1 are \(3! + 3! = 12\) permutations (1 is present in the first and second groups).

Finally, we remain with \(168 - 6 - 12 - 1 = 149\) years with the property requested.

CC145. Can a natural number be divisible by all numbers between 1 and 500 except for some two consecutive ones? If so, find these two numbers (show all possible cases).

Problem 3 of grade 8 level of 42nd Ural tournament of young mathematicians.

We received two correct solutions. We present the solution of Titu Zvonaru below.

Let \(n\) be a natural number less than 500 satisfying the statement of the problem. Let \(p\) be a number such that \(n\) is not divisible by \(p\). If \(p = a \cdot b\) with \(a, b > 1\) relatively prime, we deduce that \(n\) is not divisible by \(a\) or is not divisible by \(b\). Since \(a\) (or \(b\)) and \(p\) are not consecutive, we obtain a contradiction. This yields that \(p\) is a power of a prime. If \(p \leq 250\) then \(n\) is not divisible by \(2p\). It follows that \(p > 250\).

By the same reasoning, \(p + 1\) must be a power of a prime, and \(p + 1 > 250\). One of them is even, meaning it must be a power of 2 (the only even prime). It is not hard to see that \(256 = 2^8\) is the only number that fits our criteria. So either \(p = 256\) or \(p + 1 = 256\). If \(p = 256\) then \(p + 1 = 257\) which is easily seen to be prime. If \(p + 1 = 256\) then \(p = 255\) which factors as \(255 = 3 \cdot 5 \cdot 17\), not a power of a prime. As a conclusion, the number

\[
n = \text{lcm}(1, 2, \ldots, 254, 255, 258, 259, \ldots, 500)
\]

is divisible by all numbers between 1 and 500, except for consecutive numbers 256 and 257.