Angle Bisectors in a Triangle
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In this article, we have collected some geometric facts which are directly or tangentially related to the angle bisectors in a triangle. These results vary from easy lemmas to serious theorems, but we will not classify them; rather, we will just number them. Every statement that occurs without a proof is considered as an exercise for the reader. In fact, even the presented proofs are rather concise, which allows the reader to fill in the details.

Everyone should know this
Let us first recall some standard notation: let $ABC$ be the given triangle, $S_{ABC}$ is its area, $|BC| = a$, $|CA| = b$, $|AB| = c$, $2p = a + b + c$, $O$ and $R$ are the circumcentre and the circumradius, $I$ and $r$ are the incentre and the inradius. Furthermore, the triangle has three excircles, each of which touches one side of a triangle and the extensions of the other two sides. Their centres and their radii will be denoted by $I_a$, $I_b$, $I_c$, $r_a$, $r_b$, $r_c$ ($I_a$ denotes the centre of the excircle touching the side $BC$ and extensions of the sides $AB$ and $AC$ with $r_a$ being its radius). Further notation will be presented as needed.

1. Suppose the internal bisector of $\angle A$ intersects the side $BC$ at a point $A_1$. Then
   \[
   \frac{|BA_1|}{|A_1C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.
   \]

2. Suppose the external angle bisector of $\angle A$ intersects the line $BC$ at the point $A_2$. Then
   \[
   \frac{|BA_2|}{|A_2C|} = \frac{|BA|}{|AC|} = \frac{c}{b}.
   \]

3. $S_{ABC} = pr$.

4. $S_{ABC} = (p - a)r_a$.

5. Let $M$ be the point of tangency of the incircle with the side $AB$. Then $|AM| = p - a$.

6. Let $M$ be the point of tangency of the excircle with the centre $I_a$ and the line $AB$. Then $|AM| = p$.

7. Points $B$ and $C$ lie on the circle with diameter $II_a$ and the centre of that circle lies on a circumcircle (see Figure 1.)

Therefore, the centre $I$ of the incircle has the following property: the lines $AI$, $BI$ and $CI$ (that is, the angle bisectors of the triangle) go through the centres of the circumcircles of triangles $BIC$, $CIA$ and $AIB$, respectively. The converse is true as well, namely:

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8. If the lines $AM$, $BM$ and $CM$ go through the centres of the circumcircles of triangles $BMC$, $CMA$ and $AMB$, then $M$ is the centre of the incircle of $ABC$.

Indeed, let $M_a$, $M_b$ and $M_c$ be the points of intersection (different from $M$) of the lines $AM$, $BM$ and $CM$ with the corresponding circles (see Figure 2). Then $MM_a$, $MM_b$ and $MM_c$ are diameters of those circles; therefore, $M_aA$, $M_bB$ and $M_cC$ are altitudes of the triangle $M_aM_bM_c$. This implies that $\angle BAM = \angle BM_aC = \angle CM_bM = \angle CAM$, which means that $M$ lies on the angle bisector of angle $A$ and, analogously, on the angle bisectors of angles $B$ and $C$.

Distances between centres of special circles

9. $|OI|^2 = R^2 - 2Rr$ (Euler's formula).

10. $|OI_a|^2 = R^2 + 2Rr_a$.

11. $|II_a|^2 = 4R(r_a - r)$.

For the proof of 9 and 10, recall that if $M$ and $N$ are points of intersection of a line passing through an arbitrary point $P$ with the circle of radius $R$ and centre $O$, then $|PM| \cdot |PN| = |R^2 - |OP|^2|$; this follows from the similarity of triangles
$PMM'$ and $PNN'$, where $M'$ and $N'$ are points of intersection of the line $OP$ with the circle (see Figure 3). It implies that $R^2 - |OI|^2 = |IA| \cdot |IL|$, where $L$ is the point of intersection of the angle bisector of angle $A$ and the circumcircle (see Figure 4). But $|IA| = r/ \sin (\angle A/2)$ and, by 7, $|IL| = |LB| = 2R \times \sin (\angle A/2)$, so $R^2 - |OI|^2 = 2Rr$. Analogously,

$$|OI_a|^2 - R^2 = |I_aL| \cdot |I_aA| = 2R \times \sin (\angle A/2) \times \frac{r_a}{\sin (\angle A/2)} = 2Rr_a.$$ 

Finally,

$$|II_a|^2 = 2|IL| \cdot (|I_aA| - |IA|) = 4R \sin (\angle A/2) \cdot \frac{r_a - r}{\sin (\angle A/2)} = 4R(r_a - r).$$

**Figure 4: See Problems 9 and 10.**

12. Consider the points symmetric to the centres of the excircles with respect to the centre of the circumcircle. These points lie on the circle of radius $2R$ with the centre $I$.

**Two extremal properties of the centre of the incircle**

Consider an arbitrary point $M$ inside the triangle $ABC$. There are many inequalities concerning the distances between $M$ and the vertices of the triangle. We will consider two such inequalities.

13. Let $A_1$ be the point of intersection of the line $AM$ and the circumcircle. Then

$$\frac{|BM| \cdot |CM|}{|A_1M|} \geq 2r$$

and equality holds if $M$ coincides with $I$.

Suppose that the smallest value of $f(M) = \frac{|BM| \cdot |CM|}{|A_1M|}$ is achieved when $M$ is some point inside $ABC$. We will show that $M = I$. Then, since $f(I) = 2r$ (this follows, for example, from similar triangles $BID$ and $I_aIC$ in Figure 4), it would imply...
that if \( f(M) \) achieves its minimum inside \( ABC \), then \( f(M) \geq 2r \). The italicized statement is far from trivial and should be carefully proven.

Construct the circumcircle of triangle \( AMC \) (see Figure 5). Consider triangles \( CMA_1 \) formed by moving the point \( M \) along the arc \( AC \) — they are all similar (why?) and hence the ratio \( |CM|/|A_1M| \) is constant for all of them. Therefore, if the minimum of \( f(M) \) is achieved at \( M \), then the line \( BM \) must go through the circumcentre of the triangle \( AMC \) (otherwise, we could reduce \( |BM| \) while keeping \( |CM|/|A_1M| \) constant). Now, let \( B_1 \) and \( C_1 \) be the points of intersection of lines \( BM \) and \( CM \) with the circumcircle of \( ABC \). Then, as we saw in the proof of 9, we have \( |MA| \cdot |MA_1| = |MB| \cdot |MB_1| = |MC| \cdot |MC_1| \) and hence

\[
\frac{|BM| \cdot |CM|}{|A_1M|} = \frac{|CM| \cdot |AM|}{|B_1M|} = \frac{|AM| \cdot |BM|}{|C_1M|}.
\]

Therefore, lines \( AM \) and \( CM \) must also pass through the circumcentres of triangles \( BMC \) and \( AMB \) respectively. Then \( M \) is the circumcentre of \( ABC \) (by 8).

![Figure 5: See Problem 13.](image1)

![Figure 6: See Problem 14.](image2)

One must always be careful when using indirect proofs like the one above (where we did not directly prove that \( f(M) \geq f(I) \) for all points \( M \) inside \( ABC \)) since a function does not always achieve its minimum and maximum.

14. Show that

\[
|AM| \sin \angle BMC + |BM| \sin \angle CMA + |CM| \sin \angle AMB \leq p,
\]

and equality holds if \( M \) coincides with \( I \).

The proof of this statement will also be indirect: we will show that the point \( M \) where the left-hand side achieves its maximum (if it exists!) coincides with \( I \).

Construct the circumcircle of triangle \( BMC \) and extend the line \( AM \) until the second point of intersection \( A_2 \) (see Figure 6). Apply Ptolemy’s theorem to the quadrangle \( BMCA_2 \) to get:

\[
|BM| \cdot |A_2C| + |CM| \cdot |A_2B| = |BC| \cdot |A_2M|.
\]
Since the lengths of chords of a circle are proportional to the sines of angles subtended by those chords, we have

\[ |BM| \sin \angle A_2 MC + |CM| \sin \angle A_2 MB = |A_2 M| \sin \angle BMC \]

or

\[ |BM| \sin \angle AMC + |CM| \sin \angle AMB = |A_2 M| \sin \angle BMC. \]

Comparing the last equation to 14, we see that the left side of the inequality equals \(|AA_2| \sin \angle BMC\). Therefore, the line \(AM\) must go through the circumcentre of \(BMC\) since otherwise we can increase the value of the left-hand side of 14 by moving \(M\) along the arc \(BC\). The rest of the proof is similar to that of 13.

We leave it to the reader to prove that if \(M = I\), then \(|AA_2| \sin \angle BMC = p\). To see that, you can use 6 and 7 and the fact that \(\angle BIC = 90^\circ + \angle A/2\).

**When intuition fails**

When two similar elements of a triangle are equal (such as two angles or two medians), it seems natural to expect the triangle to be isosceles. Among the problems of this type, one of the hardest to prove is the Steiner-Lehmus theorem.

**15.** If a triangle has two angle bisectors of equal lengths, then it is isosceles.

This problem is well-known, whereas the following amusing variation is not usually familiar even to geometry buffs.

**16.** Suppose a triangle \(ABC\) has angle bisectors \(AA_1, BB_1\) and \(CC_1\). If the triangle \(A_1B_1C_1\) is isosceles, is \(ABC\) isosceles as well?

Experiments with the graphic software suggest that the answer is no; \(\triangle A_1B_1C_1\) can be isosceles while \(\triangle ABC\) is not. We do not know of any brief, elegant construction of a counterexample.

And for now, some more problems.

**17.** Prove that the angle bisector in a triangle bisects the angle between the circumradius and the altitude from the same vertex.

**18.** Let \(AA_1\) be the angle bisector of \(\angle A\) in a triangle \(ABC\). Show that

\[ |AA_1| = \sqrt{bc - |BA_1| \cdot |CA_1|} = \frac{2bc \cos (\angle A/2)}{b + c}. \]

**19.** Suppose a triangle \(ABC\) has angle bisectors \(AA_1, BB_1\) and \(CC_1\). Show that the altitudes of \(ABC\) are angle bisectors of \(A_1B_1C_1\).

**20.** Let \(M\) and \(N\) be the projections of the point of intersection of the altitudes of \(ABC\) onto the internal and external angle bisector of angle \(A\). Show that the line \(MN\) divides the side \(BC\) in half.

**21.** Let \(S\) be the sum of the areas of the three triangles whose vertices are the points where an excircle touches the sides (or their extensions) of the given triangle.
Let $T$ be the area of the triangle formed by the points where the incircle touches the sides of $ABC$. Prove that $S = S_{ABC} + T$.

22. Suppose a triangle $ABC$ has angle bisectors $AA_1$, $BB_1$ and $CC_1$; let $L$ and $K$ be the points of intersection of the lines $AA_1$ with $B_1C_1$ and $CC_1$ with $A_1B_1$, respectively. Show that $BB_1$ bisects angle $LBK$.

23. Let $M$ and $N$ be the midpoints of the diagonals $AC$ and $BC$ of a cyclic quadrilateral $ABCD$. Prove that if $BC$ bisects angle $ANC$, then $AC$ bisects angle $BMD$.

24. In a triangle $ABC$, let $M$ be the point of intersection of the angle bisector of angle $B$ with the line passing through the midpoint of $AC$ and the midpoint of the altitude from the vertex $B$. Let $N$ be the midpoint of the angle bisector of angle $B$. Show that the angle bisector of angle $C$ also bisects angle $MCN$.

25. Suppose a triangle $ABC$ has angle bisectors $AA_1$, $BB_1$ and $CC_1$, and construct the circle $O$ through the points $A_1, B_1$ and $C_1$. Consider the three chords of $O$ formed by the segments of the sides of $ABC$ lying inside $O$. Prove that the length of one of these chords is equal to the sum of the other two.

26. In a triangle $ABC$, let $K$ and $L$ be points on the sides $AB$ and $BC$, respectively, such that $|AK| = |KL| = |LC|$. Draw the line parallel to the angle bisector of angle $B$ through the point of intersection of the lines $AL$ and $CK$. Let $M$ be the point of intersection of this line with the line $AB$. Show that $|AM| = |BC|$.

27. Let $ABCD$ be a cyclic quadrilateral. Let $K$ be the point of intersection of the extensions of the sides $AB$ and $CD$; let $L$ be the point of intersection of the extensions of the sides $BC$ and $AD$. Show that the bisectors of the angles $BKC$ and $BLA$ are perpendicular and intersect on the line connecting the midpoints of $AC$ and $BD$.

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