No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


3961. Proposed by Michel Bataille.

In a triangle $ABC$, let $\angle A \geq \angle B \geq \angle C$ and suppose that

$$\sin 4A + \sin 4B + \sin 4C = 2(\sin 2A + \sin 2B + \sin 2C).$$

Find all possible values of $\cos A$.

We received two correct solutions and one incorrect submission. We present the solution by Kee-Wai Lau, modified by the editor.

Using $A + B + C = \pi$ and trigonometric formulas, we can show that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

In detail,

$$\sin 2A + \sin 2B + \sin 2C = \sin 2A + \sin 2B - \sin(2A + 2B)$$

$$= 2 \sin(A + B) \cos(A - B) - 2 \sin(A + B) \cos(A + B)$$

$$= 2 \sin(A + B) [\cos(A - B) - \cos(A + B)]$$

$$= 2 \sin(A + B) (-2 \sin(A) \sin(-B))$$

$$= 2 \sin C(2 \sin A \sin B)$$

$$= 4 \sin A \sin B \sin C.$$

Replace $A$, $B$ and $C$ in the above calculation by $2A$, $2B$ and $2C$ to get

$$\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C,$$

the only difference being in the penultimate line: since $2A + 2B + 2C = 2\pi$ (instead of $\pi$), we have $\sin(2A + 2B) = -\sin(2C)$, which introduces an extra minus sign.

Using these two equalities, the given relation is equivalent to

$$-4 \cos A \cos B \cos C = 1. \quad (1)$$

Since the product of cosines in (1) is negative, and $A \geq B \geq C$, we must have $A > \frac{\pi}{2} > 2C > 0$.

Using $A + B + C = \pi$, from (1) we get $4 \cos A \cos(A + C) \cos C = 1$, and so

$$4 \cos A(\cos A \cos C - \sin A \sin C) \cos C = 1.$$
SOLUTIONS /307

Divide both sides by \( \cos^2 C \) to get

\[
4 \cos^2 A - 4 \sin A \cos A \tan C = \frac{1}{\cos^2 C}.
\]  

(2)

Let \( x = \cos A \) and \( t = \tan^2 C \). From \( A > \frac{\pi}{2} > 2C > 0 \) it follows that \(-1 < x < 0\) and \(0 < t < 1\). Rewrite (2) using this notation and rearrange to get

\[
4x^2 - (1 + t) = 4x\sqrt{1 - x^2}t.
\]

(3)

Square both sides of (3) and move all the terms to one side to get

\[
16(1 + t)x^4 - 8(1 + 3t)x^2 + (t + 1)^2 = 0.
\]

Applying the quadratic formula,

\[
x^2 = \frac{1 + 3t \pm \sqrt{3t + 6t^2 - t^3}}{4(1 + t)}.
\]

We check whether squaring introduced extraneous solutions. Since \(-1 < x < 0\) and \(0 < t < 1\), \(4x^2 - (1 + t) < 0\) whenever (3) holds. For the solutions obtained from the quadratic formula we have

\[
x^2 = \frac{1 + 3t + \sqrt{3t + 6t^2 - t^3}}{4(1 + t)}, \quad (3) \text{ is not satisfied.}
\]

Therefore, since \( x < 0 \), we must have

\[
x = -\frac{1}{2} \sqrt{\frac{1 + 3t - \sqrt{3t + 6t^2 - t^3}}{1 + t}}.
\]

(4)

We want to find the range of values of \( x \) for \( t \in (0, 1) \). Implicitly differentiate \( x^2 = \frac{1 + 3t - \sqrt{3t + 6t^2 - t^3}}{4(1 + t)} \) to get

\[
\frac{dx}{dt} = \frac{4\sqrt{3t + 6t^2 - t^3} + t^3 + 3t^2 - 9t - 3}{16(1 + t)^2\sqrt{3t + 6t^2 - t^3}}.
\]

There is no \( t \in (0, 1) \) for which either \( x = 0 \) or \( 16(1 + t)^2\sqrt{3t + 6t^2 - t^3} = 0 \), so we conclude that the critical points of \( x \) satisfy

\[
4\sqrt{3t + 6t^2 - t^3} = -t^3 - 3t^2 + 9t + 3 \iff 3 + 9t - 3t^2 - t^3 - 16(3t + 6t^2 - t^3) = 0 \iff t^6 + 6t^5 - 9t^4 - 44t^3 - 33t^2 + 6t + 9 = 0 \iff (t - 3)(t+1)^3(2t^2 + 6t - 3) = 0.
\]

The only critical point in the range \([0, 1]\) is \( t = 2\sqrt{3} - 3 \). The corresponding value of \( x \), obtained after a tedious but straightforward calculation, is \( \frac{1 - \sqrt{3}}{2} \). From (4), we easily evaluate

\[
\lim_{t \to 0^+} x = -\frac{1}{2} \quad \text{and} \quad \lim_{t \to 1^-} x = -\frac{\sqrt{2} - \sqrt{3}}{2},
\]

allowing us to conclude that for \( t \in (0, 1) \) we have \(-\frac{1}{2} < x \leq \frac{1 - \sqrt{3}}{2}\).
Finally, we check that for each $x$ in this interval there is a corresponding triangle whose angles $A$, $B$ and $C$ satisfy the given relation. Suppose $x_0$ is such that $-\frac{1}{2} < x_0 < \frac{1}{2}$. Let $A = \cos^{-1}(x_0)$; since $\cos^{-1}$ is a decreasing function we have $A < \cos^{-1}(-0.5) = \frac{2\pi}{3}$. By the intermediate value theorem, since $x$ is continuous on $[0, 2\sqrt{3} - 3]$, there exists a $t_0$ in this interval such that $x_0 = x(t_0)$. Let $C = \tan^{-1}((\sqrt{3} - 3))$. Note that $\sqrt{t_0} \leq \sqrt{2\sqrt{3} - 3} < \sqrt{3}$, whence $C < \frac{\pi}{4}$. Let $B = \pi - A - C$; the earlier comments about the ranges for $A$ and $C$ imply $B > 0$.

We claim that a triangle with angles $A$, $B$ and $C$ satisfies the relation given in the problem (note : it seems likely that $B > C$ from this construction, but it is not immediately obvious, and anyway it is not needed since if $A$, $B$ and $C$ satisfy the relation but $B < C$ we can switch the labels of the vertices $B$ and $C$). From the construction, $\cos C = (1 + \tan^2 C)^{-1/2} = (1 + t_0)^{-1/2}$ and $\cos A = x_0$. Moreover, $x_0$ and $t$ satisfy equation (3). We calculate (using trig equalities to evaluate $\sin A$ and $\sin C$)

\[
\cos B = \cos(\pi - (A + C)) = -\cos(A + C) = -\cos A \cos C + \sin A \sin C = -x_0\sqrt{\frac{1}{1+t_0}} + \sqrt{1-x_0^2} \cdot \sqrt{\frac{1}{1+t_0}} = \sqrt{\frac{1}{1+t_0}}(-x_0 + \sqrt{1-x_0^2} \cdot \sqrt{t_0}).
\]

Since $x_0 \neq 0$ we can rearrange (3) to get $\sqrt{1-x_0^2} \cdot \sqrt{t_0} = x_0 - \frac{1+t_0}{4x_0}$; hence $\cos B = \sqrt{\frac{1}{1+t_0}} \cdot \frac{1+t_0}{4x_0} = \frac{\cos C(1+t_0)}{4 \cos A}$. It follows that $-4 \cos A \cos B \cos C = 1$, so $A$, $B$ and $C$ satisfy (1), which is equivalent to the equality given in the question.

Therefore, we conclude that the possible range of values for $\cos A$ is given by $-\frac{1}{2} < \cos A \leq \frac{1}{2}$.

**3962. Proposed by Michel Bataille.**

Let $ABC$ be a nonequilateral triangle, $\Gamma$ its circumcircle and $\ell$ its Euler line. Let its medians from $A, B, C$ meet $\Gamma$ again at $A_1, B_1, C_1$, respectively, and let $M = t_0 \cap t_C, N = t_C \cap t_A, P = t_A \cap t_B$ where $t_A, t_B, t_C$ are the tangents to $\Gamma$ at $A, B, C$, respectively.

Prove that the lines $MA_1, NB_1, PC_1$ and $\ell$ are concurrent or parallel and that the latter occurs if and only if $\cos A \cos B \cos C = -\frac{1}{4}$.

We received three correct submissions. We present the solution by Oliver Geupel.

Consider the problem in the plane of complex numbers where $\Gamma$ is the unit circle with centre $O$. Let $a, b, c, a_1, \ldots$ denote the complex numbers representing the respective points $A, B, C, A_1, \ldots$, and $G$ with coordinate $g = (a + b + c)/3$ denote the centroid of triangle $ABC$. Since $M$ is the intersection of the tangents from $B$ and $C$, it must be the inverse in $\Gamma$ of the midpoint $\frac{b+c}{2}$ of the chord $BC$, namely $m = \frac{2c}{b+c}$ (where $xx = 1$ is used for points of $\Gamma$). Since the midpoint of $BC$ belongs...
to the chord $AA_1$, we have \((\frac{b+c}{2} - a)(\overline{m} - \overline{a}) = (\frac{b+c}{2} - \overline{a})(a_1 - a)\), so that
\[
\left(\frac{b+c}{2} - a\right) \left(\frac{1}{a_1} - \frac{1}{a}\right) = \frac{(b+c)(b-c)}{2bc} (a_1 - a),
\]
and $a_1 = \frac{bc(2a-b-c)}{a(b+c)-2bc}$.

Observing that
\[
\cos A \cos B \cos C = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} - 1 = \frac{BC^2 + CA^2 + AB^2}{8} - 1
\]
\[
= \frac{1}{8} \left( (b-c) \left(\frac{1}{b} - \frac{1}{c}\right) + (c-a) \left(\frac{1}{c} - \frac{1}{a}\right) + (a-b) \left(\frac{1}{a} - \frac{1}{b}\right) \right) - 1
\]
\[
= \frac{a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2}{abc} - \frac{1}{4},
\]
the condition $\cos A \cos B \cos C = -\frac{1}{4}$ is equivalent to $\sum_{\text{sym}} a^2b = 0$ (where the sum consists of six terms). We consider the cases $\cos A \cos B \cos C = -\frac{1}{4}$ and $\cos A \cos B \cos C \neq -\frac{1}{4}$ in succession.

When $\cos A \cos B \cos C = -\frac{1}{4}$, a tedious but straightforward calculation leads to
\[
g(\overline{m} - \overline{a}) - g(m - a_1) = \frac{(a^2 - bc)(b-c)^2}{3(a(b+c) - 2bc)(2a-b-c)abc(b+c)} \sum_{\text{sym}} a^2b = 0,
\]
whence $\frac{m-a}{\overline{m}-\overline{a}} = \frac{g}{\overline{g}}$; that is, the lines $MA_1$ and $\ell = OG$ are parallel. Similarly, $NB_1$ and $PC_1$ are parallel to $\ell$. The first case is complete.

Next we examine the case $\cos A \cos B \cos C \neq -\frac{1}{4}$. Let $S$ be the point with coordinate $s = \frac{2abc(a+b+c)}{\sum_{\text{sym}} a^2b}$. Then,
\[
s\overline{g} = \frac{2(a+b+c)(ab+bc+ca)}{3\sum_{\text{sym}} a^2b} = \overline{g},
\]
so that the point $S$ lies on the line $\ell = OG$. A straightforward calculation shows that
\[
(m - a_1)(\overline{m} - \overline{s}) = \frac{2abc(b-c)^2}{(b+c)^2\sum_{\text{sym}} a^2b} = (\overline{m} - \overline{a_1})(m - s).
\]
Hence the points $M$, $A_1$, and $S$ are collinear. Similarly, the point $S$ lies on the lines $NB_1$ and $PC_1$. Consequently, the lines $MA_1$, $NB_1$, $PC_1$ and $\ell$ are concurrent at the point $S$.

Editor’s comments. The other solutions found, using areal coordinates with respect to $\triangle ABC$, that the point
\[
S = (a^2(b^4 + c^4 - a^4) : b^2(c^4 + a^4 - b^4) : c^2(a^4 + b^4 - c^4))
\]
lies on the lines $MA_1, NB_1, PC_1$. Fanchini identified it as the Exeter point ($X_{22}$ in Kimberling’s *Encyclopedia of Triangle Centers*), which is known to lie on the Euler line.

**3963. Proposed by D. M. Bătinețu and Neculai Stanciu.**

Let $A \in M_n(\mathbb{R})$ such that $A^2 = 0 \in M_n(\mathbb{R})$ and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$. Prove that $\det(xA + yI_n) \geq 0$.

We received eleven correct submissions. We present the solution by Matei Coiculescu.

Let $A \in M_n(\mathbb{R})$ such that $A^2 = 0 \in M_n(\mathbb{R})$ and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$.

If $y = 0$, then $x = 0$ and the inequality is satisfied trivially. If $y \neq 0$, let

$$M = \frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n \in M_n(\mathbb{R}).$$

Then,

$$M^2 = \left(\frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n\right) \cdot \left(\frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n\right) = \frac{x^2}{4y} \cdot A^2 + xA + yI_n = xA + yI_n$$

based on the hypothesis, that $A^2 = 0$. Hence,

$$\det(xA + yI_n) = \det(M^2) = (\det M)^2 \geq 0.$$ 

We make the observation that the condition $4y \geq x^2$ is not necessary, we only need $y > 0$.

**3964. Proposed by George Apostolopoulos.**

Let $P$ be an arbitrary point inside a triangle $ABC$. Let $a$, $b$ and $c$ be the distances from $P$ to the sides $BC$, $AC$ and $AB$, respectively. Prove that

$$\frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^4}{\sin^4 A + \sin^4 B + \sin^4 C} \leq 12R^2,$$

where $R$ denotes the circumradius of $ABC$. When does the equality occur?

We received four correct solutions. We present the solution by Oliver Geupel.

Let $x = BC$, $y = CA$, $z = AB$. By the Cauchy-Schwarz Inequality, we have

$$xyz(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 = (\sqrt{yz}\sqrt{xa} + \sqrt{zx}\sqrt{yb} + \sqrt{xy}\sqrt{zc})^2 \leq (xy + yz + zx)(xa + yb + zc).$$

Also note that

$$(xy + yz + zx)^2 \leq (xy + yz + zx)^2 + \sum_{cyc} \left(\frac{3}{2}(x^2 - y^2)^2 + x^2(y - z)^2\right)$$

$$= 3(x^4 + y^4 + z^4)$$

*Crux Mathematicorum*, Vol. 41(7), September 2015
as well as
\[ x = 2R \sin A, \quad y = 2R \sin B, \quad z = 2R \sin C. \]  
(3)
Let \( K \) denote the area of triangle \( ABC \). Then
\[ xyz = 4RK = 2R(xa + yb + zc). \]  
(4)
Putting (1) - (4) together, we obtain
\[
(\sqrt{a} + \sqrt{b} + \sqrt{c})^4 \leq \frac{(xy + yz + zx)^2(xa + yb + zc)^2}{(xyz)^2(\sin^4 A + \sin^4 B + \sin^4 C)}
\]
\[
\leq \frac{3(x^4 + y^4 + z^4)}{4R^2(\sin^4 A + \sin^4 B + \sin^4 C)}
\]
\[
= \frac{3}{4} \cdot \frac{x^4 + y^4 + z^4}{R^2(\sin^4 A + \sin^4 B + \sin^4 C)} \cdot R^2
\]
\[
= 12R^2.
\]
This completes the proof. The equality holds in (2) if and only if \( x = y = z \). Under the condition \( x = y = z \), the equality in (1) then holds if and only if \( a = b = c \).
Hence, the equality holds in the inequality of the problem if and only if triangle \( ABC \) is equilateral and \( P \) is its midpoint.

3965. Proposed by Ovidiu Furdui and Alina Sîntămărian.

Determine the interval of convergence of the power series
\[
\sum_{n=1}^{\infty} \left( \ln \frac{1}{2} + 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{(-1)^n}{n} \right) x^n
\]
and its value at \( x \) for each \( x \) in this interval.

We received four correct solutions and four incorrect or incomplete submissions. We present the solution by Michel Bataille.

Let \( a_n = \ln \frac{1}{2} + 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{(-1)^n}{n} \) where \( n \) is a positive integer. Applying Taylor’s formula with integral remainder to \( \ln(1 + x) \) between 0 and 1, we obtain
\[
\ln(2) = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n+1}}{n} + (-1)^n \int_0^1 \frac{(1-t)^n}{(1+t)^{n+1}} \, dt
\]
\[
= 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n+1}}{n} + (-1)^n \int_0^1 \frac{u^n}{1+u} \, du
\]
(with the help of the substitution \( t = \frac{1-u}{1+u} \)). It follows that
\[
a_n = (-1)^{n+1} \int_0^1 \frac{u^n}{1+u} \, du.
\]
Now, 
\[ |a_n| = \int_0^1 \frac{u^n}{1 + u} \, du \leq \int_0^1 \frac{u^n}{1 + u} \, du = \frac{1}{n+1} \]
for every positive integer \( n \), so that \( \lim_{n \to \infty} a_n = 0 \). Moreover, 
\[ |a_{n+1}| - |a_n| = \int_0^1 \frac{u^n(u-1)}{1 + u} \, du \leq 0, \]
so that the sequence \( (|a_n|)_{n \geq 1} \) is nonincreasing. From Leibniz's Alternating Series Test, the series \( \sum_{n=1}^{\infty} a_n \) is convergent. Thus, the radius of convergence \( R \) of the power series \( \sum_{n=1}^{\infty} a_n x^n \) satisfies \( R \geq 1 \).

On the other hand, integrating by parts, we obtain
\[ \int_0^1 \frac{u^n}{1 + u} \, du = \left[ \frac{u^{n+1}}{n+1} \cdot \frac{1}{1 + u} \right]_0^1 + \frac{1}{n+1} \int_0^1 \frac{u^{n+1}}{(1 + u)^2} \, du = \frac{1}{2(n+1)} + \frac{1}{n+1} J_n, \]
with 
\[ 0 \leq J_n = \int_0^1 \frac{u^{n+1}}{(1 + u)^2} \, du \leq \frac{1}{n+2}. \]
Hence \( \lim_{n \to \infty} J_n = 0 \), and it follows that \( \int_0^1 \frac{u^n}{1 + u} \, du \sim \frac{1}{n} \) as \( n \to \infty \). From this result, we deduce that the series \( \sum_{n=1}^{\infty} a_n (-1)^n \) is divergent and so \( R \leq 1 \).

We conclude that \( R = 1 \) and that the interval of convergence is \((-1, 1]\).

Let \( N \) be an integer such that \( N > 1 \) and \( x \in (-1, 1] \). Then,
\[ \sum_{n=1}^{N} a_n x^n = -\sum_{n=1}^{N} \int_0^1 \frac{(-ux)^n}{1 + u} \, du \]
\[ = -\int_0^1 \frac{1}{1 + u} (-ux)(1 + (-ux) + (-ux)^2 + \cdots + (-ux)^{N-1}) \, du \]
\[ = \int_0^1 \frac{ux}{(1 + u)(1 + ux)} \, du - \int_0^1 \frac{(-1)^N(ux)^{N+1}}{(1 + u)(1 + ux)} \, du. \]
Thus,
\[ \sum_{n=1}^{N} a_n x^n = x \int_0^1 \frac{u}{(1 + u)(1 + ux)} \, du + (-1)^{N+1} K_N \]
where \( K_N = x^{N+1} \int_0^1 \frac{u^{N+1}}{(1 + u)(1 + ux)} \, du. \)
Since
\[ 0 \leq |K_N| \leq |x|^{N+1} \int_0^1 \frac{u^{N+1} \, du}{N+2} \leq \frac{1}{N+2}, \]
we have \( \lim_{N \to \infty} K_N = 0 \). Thus, if \( S(x) = \sum_{n=1}^{\infty} a_n x^n \), by letting \( N \to \infty \), (1) yields
\[ S(x) = x \int_0^1 \frac{u}{(1 + u)(1 + ux)} \, du. \]
If \(x = 1\), then

\[
S(1) = \int_0^1 \frac{u}{(1+u)^2} \, du = \int_0^1 \left( \frac{1}{1+u} - \frac{1}{(1+u)^2} \right) \, du = \ln 2 - \frac{1}{2}
\]

and if \(|x| < 1\), using \(\frac{u}{(1+u)(1+ux)} = \frac{1}{1-x} \left( \frac{1}{1+ux} - \frac{1}{1+u} \right)\), we obtain

\[
\int_0^1 \frac{u}{(1+u)(1+ux)} \, du = \frac{\ln(1+x)}{x(1-x)} - \frac{\ln 2}{1-x}
\]

and (2) gives

\[
S(x) = \frac{\ln(1+x)}{1-x} - x \ln 2.
\]

3966. Proposed by Dao Hoang Viet.

Let \(x\) and \(y\) be the legs and \(h\) the hypotenuse of a right triangle. Prove that

\[
\frac{1}{2h+x+y} + \frac{1}{h+2x+y} + \frac{1}{h+x+2y} < \frac{h}{2xy}.
\]

We received 14 correct solutions. We present four different solutions.

Solution 1, by Digby Smith.

Since \(h\) is greater than each of \(x\) and \(y\), and since \(h^2 = x^2 + y^2 \geq 2xy\), we have that

\[
h(2h + x + y) = 2h^2 + hx + hy > 4xy + 2xy = 6xy,
\]

\[
h(h + 2x + y) = h^2 + h(2x + y) \geq 2xy + (\sqrt{2xy})(2\sqrt{2xy}) = 2xy + 4xy = 6xy,
\]

and \(h(h + x + 2y) \geq 6xy\). The desired inequality follows directly.

Solution 2, by Titu Zvonaru.

Since \(h \geq \sqrt{2xy}\), \(x + y \geq 2\sqrt{xy}\), \(2x + y \geq 2\sqrt{2xy}\) and \(x + 2y \geq 2\sqrt{2xy}\), we have that the left side of the desired inequality is no greater than

\[
\frac{1}{2\sqrt{2xy} + 2\sqrt{xy}} + \frac{2}{\sqrt{2xy} + 2\sqrt{2xy}} = \frac{1}{\sqrt{2xy}} \left[ \frac{1}{2 + \sqrt{2}} + \frac{2}{3} \right]
\]

\[
< \frac{1}{\sqrt{2xy}} \leq \frac{h}{2xy}.
\]

Solution 3, by Michel Bataille, Oliver Geupel and Dan Jonsson (independently).

By the arithmetic-geometric means inequality, each of \(2h + x + y\), \(h + 2x + y\) and \(h + x + 2y\) strictly exceeds \(3\sqrt{2hxy}\). Hence

\[
\frac{1}{2h + x + y} + \frac{1}{h + 2x + y} + \frac{1}{h + x + 2y} < \frac{1}{\sqrt{2hxy}}.
\]

Copyright © Canadian Mathematical Society, 2016
By Pythagoras’ Theorem, \(4x^2y^2 \leq (x^2 + y^2)^2 = h^4\), whence
\[
\frac{1}{2hxy} \leq \frac{h^3}{8x^3y^3}.
\]
The result follows.

**Solution 4, by Šefket Arslanagić, Salem Malikić and Henry Ricardo (independently).**

By the arithmetic-harmonic means inequality,
\[
\frac{1}{2h + x + y} = \frac{1}{(h^{-1})^{-1} + (h^{-1})^{-1} + (x^{-1})^{-1} + (y^{-1})^{-1}} < \frac{1}{16}(h^{-1} + h^{-1} + x^{-1} + y^{-1}).
\]
This, along with the analogous inequalities for the other two terms, implies that the left side is less than
\[
\frac{1}{4} \left( \frac{1}{h} + \frac{1}{x} + \frac{1}{y} \right) = \frac{1}{4hxy}[xy + h(x + y)]
\]
\[
\leq \frac{1}{4hxy} \left[ \frac{x^2 + y^2}{2} + h \sqrt{2(x^2 + y^2)} \right] = \frac{1}{4hxy} \left[ \frac{h^2}{2} + h^2 \sqrt{2} \right]
\]
\[
= \frac{h}{4xy} \left[ \frac{1}{2} + \sqrt{2} \right] < \frac{h}{2xy}.
\]

**3967. Proposed by Marcel Chiriţă.**

Determine all positive integers \(a, b\) and \(c\) that satisfy the following equation :
\[
(a + b)! = 4(b + c)! + 18(a + c)!
\]

*We received eight correct submissions, all similar. We present the solution by Joseph DiMuro.*

We shall see that there are exactly two triples that satisfy the given equation: \(a = 3, b = 4, c = 2\), and \(a = 11, b = 11, c = 10\). First of all, note that \((a + b)! > (b + c)!\) and \((a + b)! > (a + c)!\). Thus, we must have both \(a > c\) and \(b > c\).

Can we have \(a > b\)? If we divide the original equation through by \((a + c)!\), we get
\[
\frac{(a + b)!}{(a + c)!} = \frac{4(b + c)!}{(a + c)!} + 18.
\]
Since \(a + b > a + c\), \(\frac{(a + b)!}{(a + c)!}\) must be an integer. Thus, \(\frac{4(b + c)!}{(a + c)!}\) must also be an integer. Because
\[
\frac{4(b + c)!}{(a + c)!} = \frac{4}{(b + c + 1) \cdots (a + c)},
\]

*Crux Mathematicorum, Vol. 41(7), September 2015*
the product in the denominator on the right must be a divisor of 4. But we are assuming here that the three integers satisfy \( a > b > c > 0 \), so \( b + c + 1 \geq 4 \) and we can have just one factor in the denominator: we must have \( b + c + 1 = a + c = 4 \). The first equality says that \( a = b + 1 \), the second that \( b + c = 3 \); together with \( b > c \) they would imply that \( a = 3 \), \( b = 2 \), and \( c = 1 \). But this combination does not satisfy the original equation: We cannot have \( a > b \).

Similarly, we can consider the possibility that \( b > a \). If we divide the original equation through by \((b + c)!\) we get

\[
\frac{(a + b)!}{(b + c)!} = 4 + \frac{18(a + c)!}{(b + c)!}.
\]

Since \( \frac{(a + b)!}{(b + c)!} \) is an integer, \( \frac{18(a + c)!}{(b + c)!} \) must also be an integer. Because

\[
\frac{18(a + c)!}{(b + c)!} = \frac{18}{(a + c + 1) \cdots (b + c)},
\]

the product in the denominator must be a divisor of 18. But now with \( b > a > c > 0 \), each factor is at least 4, and should there be two or more factors in the denominator, their product would be at least 20. So in fact, there must be just one factor in the denominator: we must have \( a + c + 1 = b + c \), and \( b + c \) must be a divisor of 18.

We will consider the possible values for \( b + c \) in turn. Note that \( b = a + 1 \) and \( a > c \); because of that, \( b \geq c + 2 \). Thus, we cannot have \( b + c = 3 \); \( b + c \) can only equal 6, 9, or 18.

If \( b + c = 6 \), then we have

\[
\frac{(a + b)!}{6!} = 4 + \frac{18}{6} = 7,
\]

so that \( a + b = 7 \). Consequently, in this case \( a = 3 \), \( b = 4 \), and \( c = 2 \), and the given equation is indeed satisfied.

If \( b + c = 9 \), then we would have

\[
\frac{(a + b)!}{9!} = 4 + \frac{18}{9} = 6.
\]

This is impossible, since the fraction on the left would then either equal 1 or be at least 10.

Similarly, we can rule out \( b + c = 18 \) because we would get

\[
\frac{(a + b)!}{18!} = 4 + \frac{18}{18} = 5,
\]

which is impossible. Thus, the only possible combination where \( b > a \) is \( a = 3 \), \( b = 4 \), \( c = 2 \).
Finally, consider the possibility that \( a = b \). Setting \( b = a \) in the original equation gives us

\[
(2a)! = 22(a + c)!
\]

\[
\frac{(2a)!}{(a + c)!} = 22
\]

\[
(a + c + 1) \cdots (2a) = 22.
\]

Each factor on the left side is at least 4, so there can be at most two such factors. But 22 cannot be written as the product of two consecutive integers. So there must be exactly one factor on the left-hand side: \( a + c + 1 = 2a = 22 \). Thus, \( a = b = 11 \) and \( c = 10 \), which also satisfies the original equation.

3968. Proposed by Michal Kremzer.

Let \( \{ a \} = a - \lfloor a \rfloor \), where \( \lfloor a \rfloor \) is the greatest integer function. Show that if \( a \) is real and \( a(a - 2\{a\}) \) is an integer, then \( a \) is an integer.

We received 17 correct solutions, all with the same approach. We present the solution of Kathleen E. Lewis.

Since \( \{ a \} = a - \lfloor a \rfloor \), we have \( a - 2\{a\} = \lfloor a \rfloor - \{ a \} \). Thus

\[
a(a - 2\{a\}) = (\lfloor a \rfloor + \{ a \})(\lfloor a \rfloor - \{ a \}) = \lfloor a \rfloor^2 - \{ a \}^2.
\]

Since \( \lfloor a \rfloor^2 \) is an integer, \( a(a - 2\{a\}) \) can only be an integer if \( \{ a \}^2 \) is also an integer. But \( 0 \leq \{ a \} < 1 \) implies \( 0 \leq \{ a \}^2 < 1 \). Therefore, if \( a - 2\{a\} \) is an integer, then \( \{ a \} \) must be zero and thus \( a \) must be an integer.

3969. Proposed by Marcel Chiriță.

Determine the functions \( f : (\frac{8}{9}, \infty) \mapsto \mathbb{R} \) continuous at \( x = 1 \) such that

\[
f(9x - 8) - 2f(3x - 2) + f(x) = 4x - 4 + \ln \left( \frac{9x^2 - 8x}{(3x - 2)^2} \right)
\]

for all \( x \in \left( \frac{8}{9}, \infty \right) \).

We received four correct solutions and one incomplete submission. We present two solutions.

Solution 1, by Arkady Alt.

We have the following

\[
f(9x - 8) - 2f(3x - 2) + f(x)
\]

\[
= 4x - 4 + \ln \left( \frac{9x^2 - 8x}{(3x - 2)^2} \right)
\]

\[
= (9x - 8) - 2(3x - 2) + x + \ln x + \ln(9x - 8) - 2 \ln(3x - 2),
\]

so \( g(9x - 8) - 2g(3x - 2) + g(x) = 0 \), where \( g(x) := f(x) - \ln x - x \). Obviously

\[
g : \left( \frac{8}{9}, \infty \right) \mapsto \mathbb{R}
\]

is continuous at \( x = 1 \).
Let \( y := 9x - 8 \), then \( 3x - 2 = \frac{x^2}{3} \) and the original functional equation becomes

\[
g(y) - 2g \left( \frac{y + 2}{3} \right) + g \left( \frac{y + 8}{9} \right) = 0.
\] (1)

Consider the sequence \( (x_n)_{n \geq 0} \) defined recursively by \( x_{n+1} = \frac{x_n + 2}{3}, n \geq 0 \) with initial condition \( x_0 := x \), where \( x \in \left( \frac{8}{9}, \infty \right) \) and \( x \neq 1 \). Then \( x_n \in \left( \frac{8}{9}, \infty \right), n \geq 0 \) and by replacing \( y \) in (1) with \( x_n \) we obtain

\[
g(x_n) - 2g \left( \frac{x_n + 2}{3} \right) + g \left( \frac{x_n + 8}{9} \right) = 0
\]
i.e.

\[
g(x_n) - 2g(x_{n+1}) + g(x_{n+2}) = 0, \quad n \geq 0.
\]

Since \( g(x_n) - g(x_{n+1}) = g(x_{n+1}) - g(x_{n+2}) \) for any \( n \geq 0 \) then by induction

\[
g(x_n) - g(x_{n+1}) = g(x_0) - g(x_1).
\]

On the other hand, since

\[
x_{n+1} = \frac{x_n + 2}{3} \iff x_{n+1} - 1 = \frac{1}{3} (x_n - 1), \quad n \geq 0,
\]
then

\[
x_n - 1 = \frac{1}{3^n} (x_0 - 1) \iff x_n = \frac{x - 1}{3^n} + 1.
\]

Therefore, (by continuity in \( x = 1 \)) we have \( \lim_{n \to \infty} g(x_n) = g(\lim_{n \to \infty} x_n) = g(1) \).

Hence,

\[
g(x) - g \left( \frac{x + 2}{3} \right) = g(x_0) - g(x_1) = \lim_{n \to \infty} (g(x_n) - g(x_{n+1})) = g(1) - g(1) = 0.
\]

Since \( g(1) - g \left( \frac{14}{3} \right) = 0 \), then for any \( x \in \left( \frac{8}{9}, \infty \right) \) we have \( g(x) - g \left( \frac{x + 2}{3} \right) = 0 \) and therefore,

\[
g(x_n) - g \left( \frac{x_n + 2}{3} \right) = 0 \iff g(x_n) = g(x_{n+1}), \quad n \geq 0.
\]

Since \( g(x_n) = g(x_0) = g(x) \) and \( \lim_{n \to \infty} g(x_n) = g(1) \) we obtain \( g(x) = g(1) \) for any \( x \in \left( \frac{8}{9}, \infty \right) \). Therefore, \( f(x) = \ln x + x + c \), where \( c \) is any real constant.

**Solution 2, by Digby Smith.**

**Lemma.** Let \( g \) be a function satisfying \( g(0) = 0 \) which is continuous at \( t = 0 \) such that for \( t \in (-\frac{1}{3}, \infty) \), the following equation holds:

\[
g(9t) - 2g(3t) + g(t) = 0.
\]
Then $g(t) = 0$ for all $t \in (-\frac{1}{9}, \infty)$.

**Proof.** Let $s \in (-\frac{1}{9}, \infty)$. For all $j \in \mathbb{N}$ with $j \geq 2$, the following holds:

$$g \left( \frac{9 \cdot s}{3^j} \right) - 2g \left( \frac{3 \cdot s}{3^j} \right) + g \left( \frac{s}{3^j} \right) = 0.$$ 

It then follows for $n \in \mathbb{N}$ with $n \geq 2$ that

$$\sum_{j=2}^{n} \left[ g \left( \frac{9 \cdot s}{3^j} \right) - 2g \left( \frac{3 \cdot s}{3^j} \right) + g \left( \frac{s}{3^j} \right) \right] = 0,$$

$$g(s) - g \left( \frac{s}{3} \right) - g \left( \frac{s}{3^{n-1}} \right) + g \left( \frac{s}{3^n} \right) = 0,$$

$$g(s) - g \left( \frac{s}{3} \right) = g \left( \frac{s}{3^{n-1}} \right) - g \left( \frac{s}{3^n} \right).$$

Since $g$ is continuous at $t = 0$, it then follows that

$$g(s) - g \left( \frac{s}{3} \right) = \lim_{n \to \infty} \left( g \left( \frac{s}{3^{n-1}} \right) - g \left( \frac{s}{3^n} \right) \right) = g(0) - g(0) = 0,$$

so $g(s) = g \left( \frac{s}{3} \right)$. It then follows for $m \in \mathbb{N}$ that

$$g(s) = g \left( \frac{s}{3} \right) = g \left( \frac{s}{3^2} \right) = \ldots = g \left( \frac{s}{3^m} \right).$$

Again since $g$ is continuous at $t = 0$, it follows that

$$g(s) = \lim_{m \to \infty} g \left( \frac{s}{3^m} \right) = g(0) = 0.$$

That is, $g(t) = 0$ for all $t \in (-\frac{1}{9}, \infty)$.

Let $f(x) = x + \ln(x) + k + g(x - 1)$ with $k \in \mathbb{R}$ and $g(t)$ continuous at $t = 0$ such that $g(0) = 0$ satisfy the functional equation

$$f(9x - 8) - 2f(3x - 2) + f(x) = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x - 2)^2}.$$

Substituting, we get that $f$ satisfies the equation if and only if

$$g(9(x - 1)) - 2g(3(x - 1)) + g(x - 1) = 0$$

for all $x \in (\frac{8}{9}, \infty)$. Applying the Lemma, it then follows that $g(x - 1) = 0$ for all $x \in (\frac{8}{9}, \infty)$, which gives $f(x) = x + \ln(x) + k$.

**Editor’s Comments.** Bataille noticed that $\frac{17}{18} > \frac{8}{9}$ while $9 \cdot \frac{17}{18} - 8 = \frac{1}{2} < \frac{8}{9}$ so that $f \left( 9 \cdot \frac{17}{18} - 8 \right)$ is not defined! The intended version of the problem seems to be:

Determine the functions $f : (0, \infty) \to \mathbb{R}$ continuous at $x = 1$ such that

$$f(9x - 8) - 2f(3x - 2) + f(x) = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x - 2)^2} \quad \text{for all} \ x \in \left( \frac{8}{9}, \infty \right).$$

*Crux Mathematicorum*, Vol. 41(7), September 2015
3970. Proposed by Nermin Hodžić and Salem Malikić.

Let $a,b,c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{1}{10a^3 + 9} + \frac{1}{10b^3 + 9} + \frac{1}{10c^3 + 9} \geq \frac{3}{19}.
\]

We received eight correct solutions. Two submissions were incorrect and two incomplete. We present two solutions.

Solution 1, by Nermin Hodžić and Salem Malikić.

Without loss of generality, we may assume that $a + b \geq 2$. Then the following inequality holds:
\[
\frac{1}{10a^3 + 9} + \frac{1}{10b^3 + 9} \geq \frac{8}{5(a + b)^3 + 36}.
\]
To see this, note that the numerator of the difference between the left and right sides is equal to $10$ times
\[
(a - b)^2[5(a + b)^4 + 5ab(a^2 + b^2) + 30a^2b^2 - 27(a + b)]
\geq (a - b)^2[40(a + b) - 27(a + b) + 5ab(a^2 + b^2) + 30a^2b^2] \geq 0,
\]
with equality if and only if $a = b$.

Thus, the difference between the two sides of the required inequality is not less than
\[
\frac{8}{5(c - 3)^2 + 36} + \frac{1}{10c^3 + 9} - \frac{3}{19} = \frac{30c(c - 1)^2(5c^3 - 35c^2 + 60c + 36)}{19(5(3 - c)^3 + 36)(10c^3 + 9)}.
\]
Since $c \leq 1$, this quantity is nonnegative. Equality occurs when $c = 0$ or $c = 1$.

Thus, the desired inequality holds with equality exactly when
\[
(a, b, c) = (1, 1, 1), \left(\frac{3}{2}, \frac{3}{2}, 0\right), \left(\frac{3}{2}, 0, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}, 0\right).
\]

Solution 2, by Madhav Modak.

Let $E$ be equal to the left side of the desired inequality. We may assume that $a \geq b \geq c \geq 0$. The function
\[
f(x) = \frac{1}{10x^3 + 9},
\]
is concave on $[0, \alpha]$ and convex on $[\alpha, 3]$, where $\alpha = \sqrt[3]{9/20}$.

The tangent to the graph of $f$ at $(1, 1/19)$ has equation $y = g(x)$ with
\[
g(x) = -\frac{30}{19^2}(x - 1) + \frac{1}{19} = \frac{49 - 30x}{19^2}.
\]
Then
\[ f(x) - g(x) = \frac{10(x - 1)^2(30x^2 + 11x - 8)}{19^2(10x^3 + 9)} \geq 0, \]
for \( x \geq 1/2 \). Therefore, when \( c \geq 1/2 \),
\[ E \geq \frac{30}{19^2}[(1 - a) + (1 - b) + (1 - c)] + \frac{3}{19} = \frac{3}{19}. \]
Equality occurs if and only if \( a = b = c = 1 \).

The tangent to the graph of \( f \) at \((3/2, 4/171)\) has equation \( y = h(x) \) with
\[ h(x) = -\frac{40}{3 \times 19^2} \left( x - \frac{3}{2} \right) + \frac{4}{9 \times 19} = \frac{256 - 120x}{9 \times 19^2}. \]

When \( 0 \leq x \leq 3 \),
\[ f(x) - h(x) = \frac{20(x - \frac{3}{2})^2(60x^2 + 52x + 21)}{(9 \times 19^2)(10x^3 + 9)}. \]
Equality occurs if and only if \( x = 3/2 \).

Therefore,
\[ E \geq \frac{512 - 120(3 - c)}{9 \times 19^2} + \frac{1}{10c^3 + 9}. \]
When \( 0 \leq c \leq 1/2 \), we find that
\[ E - \frac{3}{19} \geq \frac{10c(120c^3 - 361c^2 + 108)}{(9 \times 19^2)(10c^3 + 9)} \geq 0, \]
with equality if and only if \( c = 3/2 \).

Thus, we obtain the desired inequality with the same conditions for equality as before.

Editor’s comments. Fanchini applied Muirhead’s Theorem to the numerator of the difference between the two sides to obtain the result, but as the argument uses an advanced result and the execution is tedious but straightforward, we do not present it here.