CC179. Matthew creates a sequence of numbers starting from the number 7. Every number in his sequence is the sum of the digits of the square of the previous number, plus 1. For example, the second number in his sequence is 14 because $7^2 = 49$ and $4 + 9 + 1 = 14$. What is the 1000th number of Matthew’s sequence?

CC180. The pages of a book are numbered 1, 2, 3, . . . A digit that appears in the number of the last page appears 20 times in the set of page numbers of the book. If the book had thirteen pages less, then the same digit would have been used 14 times in total. How many pages does the book have?

CONTEST CORNER

SOLUTIONS


CC126. Anna and Ben decided to visit Archipelago with 2009 islands. Some islands are connected by boats which run both ways. Anna and Ben are playing the following game during the trip: Anna chooses the first island on which they arrive by plane; then Ben chooses the next island to visit. Thereafter, they take turns choosing a new island. When they arrive at an island connected only to islands they have already visited, whoever’s turn it is to choose the next island loses. Prove that Anna can always win, no matter how Ben plays and how islands are connected.

Originally from 2009 Tournament of Towns, Spring, A-level, Juniors.

We received no solutions. We present a solution based on the official solution.

We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices. Anna chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will mark these edges.
Since the number of vertices is odd, there is at least one vertex which is not incident with a marked edge. Anna will start the tour there.

Suppose Ben has a move. It must take the tour to a vertex incident with a marked edge. Otherwise, Anna could have marked one more edge. Anna simply continues the tour by following that marked edge. If Ben continues to go to vertices incident with marked edges, Anna will always have a ready response. We show that Ben cannot manage to get to a vertex not incident with a marked edge. Suppose otherwise.

Consider the tour so far. Both the starting and the finishing vertices are not incident with marked edges. In between, the edges are alternately marked and unmarked. If Anna interchanges the marked and unmarked edges on this tour, she could have obtained a larger independent set of edges, a contradiction.

**CC127.** In the Great Hall of Camelot there is the Round Table with \( n \) seats. Merlin summons \( n \) knights of Camelot for a conference. Every day, he assigns seats to the knights. From the second day on, any two knights who become neighbours may switch their seats unless they were neighbours on the first day. If the knights manage to sit in the same cyclic order as on one of the previous days, the next day the conference ends. What is the maximal number of days of the conference Merlin can guarantee?

*Originally from 2010 Tournament of Towns, Fall, A-level, Juniors.*

We received no solutions. We present a solution based on the official solution.

We may assume that the knights are seated from 1 to \( n \) in cyclic clockwise order on day 1. Then seat exchanges are not permitted between knights with consecutive numbers (1 and \( n \) are considered consecutive). We will assign to each cyclic order a value called the winding number, which will be shown to be invariant under permitted seat exchanges. We obtain the winding number as follows. Merlin has \( n \) hats that he gives to the \( n \) knights. He starts by giving the first hat to knight 1, then walks clockwise around the table until he reaches knight 2, to which he gives the second hat. He continues in clockwise order giving out the remaining hats to the knights in ascending order, after which he walks on until he reaches knight 1 again. The number of times he has gone around the table is called the winding number of the cyclic order. For instance, in the following example the winding number is 4; in the first round Merlin hands out hats to knights 1 and 2, in the second round to knights 3 and 4, in the third round to knight 5, and in the fourth round to knights 6 and 7.
Next we show that the winding number is invariant under seat exchanges. If knight 1 is not involved in the seat exchange, then the hats handed out in each round remain the same. If knight 1 changes seats with knight $h$, then knight $h$ gets his hat one round earlier or later, but all the other knights get their hats in the same round as before. In both cases the winding number remains constant. Thus to guarantee that the conference lasts at least $n$ days, Merlin just has to seat the knights in $n-1$ arrangements with distinct winding number on the first $n-1$ days. As one solution, Merlin could seat the knights on the $k$-th day in the cyclic order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$. It is easily verified that this order has winding number $k$.

To show that Merlin cannot make the conference last any longer, we prove that any two cyclic orders with the same winding number can be transformed into each other via permitted seat exchanges. Since the winding number cannot be any larger than $n-1$, the knights can then force the end of the conference on the $n$-th day at the latest.

To be precise, we show that any cyclic order with winding number $k$ can be transformed into the order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$ (as a sequence of seat exchanges is reversible, we can then transform any cyclic order into any other cyclic order with the same winding number). Given such a cyclic order, we start making seat exchanges that move knight 2 counter-clockwise towards knight 1 as far as possible. If he encounters knight 3, we move both of them counter-clockwise and so on. By this we obtain a cyclic order starting with $1, h, h-1, \ldots, 2, \ldots$.

If $h = n$, we are done. Otherwise, we let knight 1 change seats with knights $h, h-1, \ldots, 3$ in turn, so that knight 2 now sits after knight 1. We repeat the process to move knight 3 to the place after knight 2, and so on, stopping when we arrive at the order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$. The following example illustrates the process.
**CC128.** In one kingdom gold sand and platinum sand are used as a currency. An exchange rate is defined by two positive integers \( g \) and \( p \); namely, \( x \) grams of gold sand are equivalent to \( y \) grams of platinum sand if \( x : y = p : g \) (\( x \) and \( y \) are not necessarily integers). At a day when \( g = p = 1001 \), the Treasury announced that on each of the following days one of the numbers, either \( g \) or \( p \) would be decreased by 1 so that after 2000 days \( g = p = 1 \). However, the exact order in which the numbers decreased is kept unknown. At the day of the announcement a banker had 1 kg of gold sand and 1 kg of platinum sand. The banker’s goal is to make exchanges so that at the end of this period he would have at least 2 kg of gold sand and 2 kg of platinum sand. Can the banker reach his goal for certain as a result of some clever exchanges?

*Originally from 2014 Tournament of Towns, Fall, A-level, Seniors.*

*We received no solutions.*

**CC129.** A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, there are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in half?

*Originally from 2013 Tournament of Towns, Fall, A-level, Seniors.*

*We received no solutions.*

**CC130.** Let \( P(x) \) be a polynomial such that \( P(0) = 1 \) and \((P(x))^2 = 1 + x + x^{100}Q(x)\), where \( Q(x) \) is also a polynomial. Prove that the coefficient of \( x^{99} \) of the polynomial \((P(x) + 1)^{100}\) is zero.

*Originally from 2014 Tournament of Towns, Spring, A-level, Seniors.*

*We received one incomplete submission.*