No. 35
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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by August 1, 2016, although late solutions will also be considered until a solution is published.

CC171. The zeroes of the polynomial \( f(x) = x^2 - ax + 2a \) are integers. What is the sum of all the possible values of the number \( a \)?

CC172. What is the area of regular hexagon \( ABCDEF \) with \( A(0,0) \) and \( C(7,1) \)?

CC173. In the following figure, an isosceles triangle with \( AB = 12 \) is divided into 4 polygons of equal area using segments perpendicular to \( AB \). Find \( x \).

CC174. Evaluate \( \sqrt{1111} - 22 \) and \( \sqrt{111111} - 2222 \). Conjecture the result for \( \sqrt{111111111111111111} - 22222222222222222222 \) and prove it.

CC175. Twenty two mathematics contests were held with five prizes given out for each one. The organizers notice that for each pair of contests, there is exactly one participant who has won a prize in both contests. Show that one of the participants has won a prize in each of the contests.

CC171. Les zéros du polynôme \( f(x) = x^2 - ax + 2a \) sont des nombres entiers. Quelle est la somme des valeurs possibles de \( a \)?

CC172. Quelle est l’aire de l’hexagone régulier $ABCDEF$ avec $A(0; 0)$ et $C(7; 1)$?

CC173. Dans la figure suivante, le triangle isocèle avec $AB = 12$ est divisé en 4 polygones de même aire en n’utilisant que des segments perpendiculaires à $AB$. Quelle est la longueur $x$?

![Diagram of an isosceles triangle divided into four polygons with a segment $x$.

CC174. Calculer $\sqrt{111} - 22$ et $\sqrt{11111} - 222$. Conjecturer le résultat de $\sqrt{111111111111111111111111} - 222222222222$ et le démontrer.

CC175. Vingt-deux concours de mathématique ont eu lieu, et pour chacun d’entre eux on a remis cinq prix. Les organisateurs s’aperçoivent alors que pour chaque paire de concours, il y a exactement un participant qui a gagné un prix dans ces deux concours. Montrer qu’un des participants a gagné un prix dans chacun des concours.

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Math Quotes

If you do not or cannot humanize the learning of mathematics through joy, shared discoveries and mutual bewilderment/wonder FIRST, then it’s not just students you will lose to disengagement. You will also lose teachers. Mathematics has become so procedural and sterile, having all the fun of the smell of antiseptics in an operating room.

At least be passionate. Leave your emotions about math tattooed all over the chalkboard. If you don’t care, then why should your students...

*By Sunil Singh.*
CONTEST CORNER
SOLUTIONS


**CC121.** Towns A and B are situated on two straight roads intersecting at the angle of $\angle ACB = 60^\circ$. One way to get from A to B is by taking the bus which goes from A to C to B; this takes 11 minutes. Alternatively, you can walk from A directly to B, which takes an hour and 10 minutes. Finally, you can first walk from A to the road on which B is situated and then take the bus to B, but this takes longer still even if the bus comes immediately.

Find the distance from A to the intersection C if you walk at the speed of 3 km/h and the bus drives at the speed of 30 km/h.

*Originally question 1 on the 1969 entrance exam to the mathematical department of the Moscow State University.*

*We received two correct solutions and one incorrect submission. We present the solution by Digby Smith, slightly modified by the editor.*

Denote the distance $AC$ by $x$, $BC$ by $y$ and $AC$ by $z$. Using the “distance = speed $\times$ time” formula, it follows that

$$z^2 = x^2 + y^2 - 2xy\cos(60^\circ) \iff$$

$$\frac{49}{4} = x^2 + \left(\frac{11}{2} - x\right)^2 - 2x\left(\frac{11}{2} - x\right) \cdot \frac{1}{2} \iff$$

$$\frac{49}{4} = x^2 + \frac{121}{4} - 11x + x^2 - \frac{11}{2}x + x^2.$$ 

Rearranging,

$$3x^2 - \frac{33}{2}x + 18 = 0 \iff$$

$$3(2x^2 - 11x + 12) = 0 \iff$$

$$(2x - 3)(x - 4) = 0.$$ 

Hence $x = 4$ km (which gives $y = \frac{3}{2}$ km), or $x = \frac{3}{2}$ km (which gives $y = 4$ km). We now need to verify the condition regarding the walking time when we combine the two modes of transportation. Let $D$ be the closest point on $CB$ to $A$. Then $AD = x\sin(60^\circ)$ and $CD = x\cos(60^\circ)$. The time needed to walk the distance $AD$ and take the bus from $D$ to $B$ is (in minutes)

$$20AD + 2DB = 10\sqrt{3} \cdot x + 2DB.$$ 

Whether $DB = CD - CB = \frac{x}{2} - y$ or $DB = CB - CD = y - \frac{x}{2}$ depends on whether $D$ is closer to $C$ than $B$ or not. We consider the two possible values of $x$ calculated above.
If \( x = 4 \) then \( CD \) is longer than \( CB \) (\( \triangle ABC \) is obtuse). That is,

\[
DB = CD - CB = 2 - \frac{3}{2} = \frac{1}{2},
\]

and the total time to walk from \( A \) to \( D \) and then take the bus to \( B \) is \( 40\sqrt{3} + 1 \) minutes. It is easy to see that this is greater than the 70 minutes needed to walk from \( A \) to \( C \) and then from \( C \) to \( B \).

If \( x = \frac{3}{2} \) then \( CD \) is shorter than \( CB \) (\( \triangle ABC \) is acute). That is,

\[
DB = CB - CD = 4 - \frac{3}{4} = \frac{13}{4},
\]

and the total time to walk from \( A \) to \( D \) and then take the bus to \( B \) is \( 15\sqrt{3} + \frac{13}{2} \) minutes. This is easily seen to be less than \( 15 \cdot 2 + \frac{13}{2} = 37 \), which is much less than the 70 minutes needed to walk from \( A \) to \( C \) and then from \( C \) to \( B \).

Hence only \( x = 4 \) satisfies the additional mixed mode of transportation time constraint. We conclude that the distance from \( A \) to \( C \) must be 4km.

**CC122.** The sequence \( \{x_n\} \) is given by the following recursion formula:

\[
x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2, \quad 0 < a < 1.
\]

Find the limit of the sequence.

*Originally question 12 on the 1977 entrance exam to the Moscow Physics Institute.*

*We received seven correct solutions and one incomplete submission. We present the solution by Henry Ricardo.*

Using mathematical induction, we show that \( x_n < x_{n+1} < 1 \) for every positive integer \( n \), thereby proving that the sequence is monotonic and bounded and hence has a limit.

When \( n = 1 \), we see that

\[
x_1 = \frac{a}{2} < \frac{a}{2} + \frac{(a/2)^2}{2} = x_2 < \frac{1}{2} + \frac{1}{8} < 1.
\]

Now suppose that \( x_N < x_{N+1} < 1 \) for some \( N \geq 2 \). Then, since \( 0 < a < 1 \),

\[
x_{N+1} = \frac{a}{2} + \frac{x_N^2}{2} < \frac{a}{2} + \frac{x_{N+1}^2}{2} < \frac{a}{2} + \frac{1}{2} < 1.
\]

and the induction is complete.

If \( \lim_{n \to \infty} x_n = L \), then the recursion formula gives us \( L = \frac{a}{2} + \frac{L^2}{2} \), or \( L^2 - 2L + a = 0 \), yielding \( L = 1 \pm \sqrt{1 - a} \). Since \( x_n < 1 \) for all \( n \), we conclude that \( L = 1 - \sqrt{1 - a} \).
CC123. Find how many pairs of integers \((x, y)\) satisfy the inequality
\[ 2^{x^2} + 2^{y^2} < 2^{1976}. \]

*Originally question 5 on the 1976 entrance exam to the All-republican Distance Education Moscow School.*

*There were no correct submissions for this problem and one incorrect submission.*

CC124. In a chess tournament, every participant played every other participant exactly once. In each game each participant scored 1 point for the win, 0.5 points for the tie and 0 points for the loss. At the end of the tournament, you discovered that in any group of any three participants there is one who, in the games against the other two, got 1.5 points. What is the maximum possible number of participants the tournament could have had?

*Originally question 3 on the 1999 entrance exam to the mathematical-mechanical department of the Belorussian State University.*

*We received no solutions to this problem.*

CC125. Orthogonal projections of a triangle \(ABC\) onto two perpendicular planes are equilateral triangles with side length 1. If the median \(AD\) of triangle \(ABC\) has length \(\sqrt{\frac{7}{8}}\), find \(BC\).

*Originally question 4 on the 1969 entrance exam to the mathematical-mechanical department of Moscow State University.*

*We received no solutions to this problem.*