OLYMPIAD SOLUTIONS


OC166. Let \(\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}\). Find the maximum value of
\[
\sum_{n=1}^{10} (na_n^2 - na_n).
\]

Originally problem 3 from day 2 of the 2012 Korea National Olympiad.
No solutions were submitted.

OC167. Find all functions \(f : \mathbb{R} \to \mathbb{R}\) such that
\[
(x - 2)f(y) + f(y + 2f(x)) = f(x + yf(x))
\]
for all \(x, y \in \mathbb{R}\).

Originally problem 2 from day 1 of the 2012 Spain National Olympiad.
We received two correct solutions. We present the solution by Joseph Ling.

It is easy to verify that (1) \(f(x) = 0\) for all \(x\) and (2) \(f(x) = x - 1\) for all \(x\) are solutions to
\[
(x - 2)f(y) + f(y + 2f(x)) = f(x + yf(x)).
\]

(*)

We claim that there are no others.
Suppose that \(f\) is not identically zero. We first observe that \(f(2) \neq 0\). For if \(f(2) = 0\), then letting \(x = 2\) in (*) , we arrive at \(f(y) = 0\) for all \(y\). Next, we observe that \(x = 1\) is the only number that can possibly be a root of \(f(x) = 0\). For if \(f(x_0) = 0\), then \(x_0 \neq 2\), and letting \(x = x_0\) in (*), we arrive at \((x_0 - 1)f(y) = 0\) for all \(y\). Since \(f\) is not identically zero, this implies that \(x_0 = 1\).

Suppose now \(x\) is a number such that \(x \neq 2\) and \(f(x) \neq 1\). Then letting
\[
y = \frac{x - 2f(x)}{1 - f(x)},
\]
we have
\[
y + 2f(x) = x + yf(x),
\]
and so, (*) is reduced to \((x - 2)f(y) = 0\). Since \(x \neq 2\), \(f(y) = 0\), and so, \(y = 1\).
It follows that \(f(x) = x - 1\).

We now prove that the above condition \(f(x) \neq 1\) is redundant by showing that \(x \neq 2 \implies f(x) \neq 1\). Suppose \(x\) is a number such that \(x \neq 2\) and \(f(x) = 1\). We
will derive a contradiction. Letting \( y = 0 \) in (*) and noting that \( f(x) = 1 \), we get a unique candidate

\[
x = x^* = 2 + \frac{1 - f(2)}{f(0)}.
\]

Thus, with the possible exceptions of \( x = 2 \) and \( x = x^* \), we always have \( f(x) = x - 1 \). In particular, letting \( y \) be any number such that none of \( y, y+2 \), and \( x^* + y \) coincide with \( 2 \) or \( x^* \), we have \( f(y) = y - 1 \), \( f(y+2) = y + 1 \), and \( f(x^* + y) = x^* + y - 1 \). Using these and the assumption that \( f(x^*) = 1 \), we let \( x = x^* \) and simplify (*) into \((x^* - 2)(y - 2) = 0\). Since \( y \neq 2, x^* = 2 \), a contradiction. This completes the proof that no number \( x \) can satisfy \( x \neq 2 \) and \( f(x) = 1 \) simultaneously.

It follows that \( x = 2 \) is the only possible exception to the rule \( f(x) = x - 1 \).

Recall that \( f(2) \neq 0 \). Choosing \( y \) to be such that none of \( y, y+2f(2), \) and \( 2+yf(2) \) coincide with \( 2 \), we have \( f(y+2f(2)) = y + 2f(2) - 1 \) and \( f(2+yf(2)) = 1+yf(2) \). Using these and letting \( x = 2 \) in (*), we arrive at \((f(2) - 1)(y - 2) = 0\). Since \( y \neq 2, f(2) = 1 = 2 - 1 \). This means that \( f(x) = x - 1 \) with no exceptions.

**OC168.** Let \( ABCD \) is a square. Find the locus of points \( P \) in the plane, different from \( A, B, C, D \) such that

\[
\angle APB + \angle CPD = 180^\circ.
\]

*Originally problem 5 from the 2012 Italy Math Olympiad.*

*We received four correct solutions and one incorrect submission. We present the solution by Michel Bataille.*

Let \( \Gamma \) be the circumcircle of the square \( ABCD \). We show that the required locus is the union of the line segments \( AC, BD \) and the short arcs \( \hat{AB} \) and \( \hat{CD} \) of \( \Gamma \) (the points \( A, B, C, D \) being excluded).

We first show that any point \( P \) of these sets satisfies the condition \( \angle APB + \angle CPD = 180^\circ \). As a chord of \( \Gamma \), a side of the square subtends an angle of \( 135^\circ \) if the vertex is on the short arc and of \( 45^\circ \) if the vertex is on the long arc. It follows
that for a point $P$ on either of the short arcs $\widehat{AB}, \widehat{CD}$, we have $\angle APB + \angle CPD = 45^\circ + 135^\circ = 180^\circ$.

Let $P$ be on the line segments $AC$ or $BD$. Without loss of generality, we may suppose that $P$ is on $A\Omega$, where $\Omega$ is the centre of the square. Let $t = \angle PBD = \angle PDB$. Then, $\angle CPD = \angle \Omega PD = 90^\circ - t$ and $\angle APB = 180^\circ - \angle BP\Omega = 180^\circ - (90^\circ - t) = 90^\circ + t$. Thus, $\angle APB + \angle CPD = 180^\circ$.

Conversely, suppose that $P$ satisfies $\angle APB + \angle CPD = 180^\circ$. If $\angle APB = \angle CPD = 90^\circ$, then $P = \Omega$, a point of the diagonals. From now on, we suppose that $\angle APB$ and $\angle CPD$ are different from $90^\circ$. One of these angles then is obtuse, say $\angle APB > 90^\circ$. The point $P$ is interior to the circle with diameter $AB$ and clearly not on the side $AB$. We distinguish the two cases when in addition $P$ is interior to the square or not.

Suppose that $P$ is not interior to the square. Let $P'$ be such that $APP'D$ is a parallelogram and $P''$ the reflection of $P'$ in $CD$. Then, $\angle DP''C = \angle APB$, hence $\angle DP''C + \angle CPD = 180^\circ$. Therefore $C, D, P, P''$ are concyclic. On the other hand, the triangles $APB$ and $DP''C$ are congruent and so $DP'' = AP$. Since $AD$ and $PP''$ are parallel, the quadrilateral $APP''D$ is an isosceles trapezium.

As such, it has a circumcircle and so $A$ is on the circle through $C, D, P, P''$. Thus,
this circle is the circumcircle $\Gamma$ of $ABCD$ and we conclude that $P$ is on the short arc $AB$ of $\Gamma$.

Suppose that $P$ is interior to the square. Again, let $P'$ be such that $APP'D$ is a parallelogram. Then, $\angle DP'C = \angle APB$, hence $\angle DP'C + \angle CPD = 180^\circ$. Therefore $C, D, P, P'$ are concyclic. It follows that $\angle DPP' = \angle DCP = \alpha$ where $\alpha = \angle APB$. Since $AD$ and $PP'$ are parallel, we deduce $\angle ADP = \angle DPP' = \alpha$.

Now, let $\theta = \angle APB$ and $\theta' = \angle APD$.

If $a$ denotes the side of the square, we have $\frac{a}{\sin \theta'} = \frac{AP}{\sin \alpha} = \frac{a}{\sin \theta}$, hence $\theta' = \theta$ or $\theta' = 180^\circ - \theta$. In the former case, we deduce $\angle PAD = \angle PAB$, hence $P$ is on $AC$. In the latter case, we have $\angle BPC = \theta$ (since $\angle BPC + \angle APD = 180^\circ$ as well) and in a similar way, $P$ is on $BD$. The proof is complete.

**OC169.** Find all positive integers $n \geq 2$ such that for all integers $0 \leq i, j \leq n$ the numbers $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have same parity.

*Originally problem 1 from the 2012 Iran National Math Olympiad TST.*

*We received no solutions.*

**OC170.** Let $ABC$ be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments $BC$, respectively $AC$ at $D$, respectively $E$. Prove that

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA).$$

*Originally problem 1 from day 2 of the 2012 Romanian TST.*

*We received one correct solution. We give the solution by Titu Zvonaru.*

Let $a = BC$, $b = CA$ and $c = AB$. By the Angle Bisector Theorem, we have

$$DC = \frac{ab}{b + c} \quad \text{and} \quad EC = \frac{ab}{a + c}.$$
By the Cosine Law on triangle $DEC$, we have

$$DE^2 = \frac{a^2b^2}{(b+c)^2} + \frac{a^2b^2}{(a+c)^2} - 2\frac{a^2b^2}{(a+c)(b+c)} \cos(C).$$

By the Cosine Law on triangle $ABC$,

$$DE^2 = \frac{a^2b^2}{(b+c)^2} + \frac{a^2b^2}{(a+c)^2} - 2\frac{a^2b^2}{(a+c)(b+c)} \cdot \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{ab(a^2bc + ab^2c + 3abc^2 - a^3c - b^3c - a^2c^2 - b^2c^2 + ac^3 + bc^3 + c^4)}{(a + c)^2(b + c)^2}.$$ 

Next, using the inequalities:

$$(c + \sqrt{ab})^2 \leq (c + a)(c + b),$$

$$2abc^2 \leq a^2c^2 + b^2c^2,$$

$$a^2bc + ab^2c \leq a^3c + b^3c,$$

we have that

$$DE^2 \leq \frac{ab(a^3c + b^3c + abc^2 + a^2c^2 + b^2c^2 - a^3c - b^3c - a^2c^2 + b^2c^2 + ac^3 + bc^3 + c^4)}{(a + c)^2(b + c)^2}$$

$$= \frac{ab(abc^2 + bc^3 + ac^3 + c^4)}{(a + c)^2(b + c)^2}$$

$$= \frac{abc^2(a + c)(b + c)}{(a + c)^2(b + c)^2}$$

$$\leq \frac{abc^2}{(c + \sqrt{ab})^2}.$$ 

Taking square roots, it suffices to prove that

$$\frac{c\sqrt{ab}}{c + \sqrt{ab}} \leq (3 - 2\sqrt{2})(c + 2\sqrt{ab}).$$

This inequality is true if and only if

$$c\sqrt{ab} \leq (3 - 2\sqrt{2})(c + 2\sqrt{ab})(c + \sqrt{ab})$$

$$(3 + 2\sqrt{2})c\sqrt{ab} \leq (3 + 2\sqrt{2})(3 - 2\sqrt{2})(c + 2\sqrt{ab})(c + \sqrt{ab})$$

$$3c\sqrt{ab} + 2\sqrt{2}c\sqrt{ab} \leq c^2 + 3c\sqrt{ab} + 2ab$$

$$0 \leq (c - \sqrt{2ab})^2$$

and the last inequality clearly holds thus the string of inequalities holds. Equality holds if and only if $a = b$ and $c^2 = 2a^2$, that is, triangle $ABC$ is an isosceles right angled triangle with angle $BCA$ as the right angle.