Application of Inversive Methods to
Euclidean Geometry : solutions

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In this part, we present the solutions to ten problems in Application of Inversive Methods to Euclidean Geometry by Andy Liu appearing in the previous issue (Volume 41 (3), p. 114–118). Solution to problem 1 appears in the original article.

Problem 2 (below left)
Two circles $\omega_1$ and $\omega_2$ are tangent externally to each other at $A$. A common exterior tangent touches $\omega_1$ at $P$ and $\omega_2$ at $Q$. The other common exterior tangent touches $\omega_1$ at $R$ and $\omega_2$ at $S$. Prove that the circumcircles of triangles $PAQ$ and $RAS$ are tangent to each other.

Solution to Problem 2
Invert with respect to $A$ (above right). Then $\omega_1$ and $\omega_2$ become parallel lines $P'R'$ and $Q'S'$. $PQ$ and $RS$ become circles $\omega'_3$ and $\omega'_4$, tangent to both $P'R'$ and $Q'S'$. The circumcircles of triangles $PAQ$ and $RAS$ become diameters $P'Q'$ and $R'S'$ of $\omega'_3$ and $\omega'_4$. These diameters are orthogonal to $PR$ and are therefore parallel to each other. Hence the two circumcircles are also tangent to each other.

Problem 3 (below left)
$AB$, $AC$ and $AD$ are three chords on a circle. Circles with $AB$ and $AC$ as diameters intersect at $E$, circles with $AB$ and $AD$ as diameters intersect at $F$, and circles with diameters $AC$ and $AD$ intersect at $G$. Prove that $E$, $F$ and $G$ are collinear.
Solution to Problem 3
Invert with respect to $A$ (above right). Then the original circle becomes the line $B'C'D'$. The other three circles become the lines $E'B''F'$, $E'C'G'$ and $F'G'D'$, and they are orthogonal to $AB'$, $AC$, and $AD$, respectively.

Hence $AE'B'C'$, $AB'F'D'$ and $AC'G'D'$ are cyclic quadrilaterals, so that $\angle B'E'C' = \angle B'AC'$, $\angle B'AF' = \angle B'D'F'$ and $\angle C'AG' = \angle C'D'G'$. It follows that

$$
\angle F'AG' = \angle B'AG' - \angle B'AF' = \angle B'AC' - \angle B'D'F' = \angle F'E'G'.
$$

Hence $A$, $E'$, $F'$ and $G'$ are concyclic, so that $E$, $F$ and $G$ are collinear.

**Problem 4** (below left)
Three circles $\omega_1$, $\omega_2$ and $\omega_3$ pass through $O$. $B$ is the other point of intersection of $\omega_1$ and $\omega_2$, $C$ is the other point of intersection of $\omega_2$ and $\omega_3$, and $A$ is the other point of intersection of $\omega_3$ and $\omega_1$. The tangent to $\omega_2$ at $O$ intersects $BC$ at $D$, the tangent at $O$ to $\omega_3$ intersects $CA$ at $E$, and the tangent at $O$ to $\omega_1$ intersects $AB$ at $F$. Prove that $D$, $E$ and $F$ are collinear.
Solution to Problem 4
Invert with respect to $O$ (above right). Then the three circles turn into triangle $A'B'C'$ while the tangent lines $OD$, $OE$ and $OF$ turn into themselves. Hence $OD'$, $OE'$ and $OF'$ are parallel to $B'C'$, $C'A'$ and $A'B'$ respectively. Moreover, $D'$, $E'$ and $F'$ lie on the circumcircles of triangles $OB'C'$, $OC'A'$ and $OA'B'$ respectively. Let $Q$ be the circumcentre of triangle $A'B'C'$. Then $Q$ lies on the perpendicular bisectors of $OD'$, $OE'$ and $OF'$. Hence $O$, $D'$, $E'$ and $F'$ are concyclic. It follows that $D$, $E$ and $F$ are collinear.

Problem 5 (below left)
Four circles $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ are such that $\omega_1$ and $\omega_2$ touch at $A$, $\omega_2$ and $\omega_3$ touch at $B$, $\omega_3$ and $\omega_4$ touch at $C$ and $\omega_4$ and $\omega_1$ touch at $D$. Prove that $A$, $B$, $C$ and $D$ are concyclic.

Solution to Problem 5
Invert with respect to $A$ (above right). Then $\omega_1$ and $\omega_2$ become a pair of parallel lines, tangent to $\omega_4'$ and $\omega_3'$ at $D'$ and $B'$ respectively. These two circles are tangent to each other at $C'$. Let $P'$ and $Q'$ be the centres of $\omega_4'$ and $\omega_3'$ respectively. Then $C'$ lies on $P'Q'$.

Since $C'D'$ and $C'B'$ are parallel, $\angle C'P'D' = \angle C'Q'B'$. Since $C'P' = D'P'$ and $C'Q' = B'Q'$,

$$\angle P'C'D' = \frac{1}{2}(180^\circ - \angle C'P'D') = \frac{1}{2}(180^\circ - \angle C'Q'B') = \angle Q'C'B'.$$

Hence $C'$ also lies on $B'D'$, which means that $A$, $B$, $C$ and $D$ are concyclic.

Problem 6 (below left)
Four circles $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ are such that $\omega_1$ and $\omega_2$ intersect at $A_1$ and $A_2$, $\omega_2$ and $\omega_3$ intersect at $B_1$ and $B_2$, $\omega_3$ and $\omega_4$ intersect at $C_1$ and $C_2$, and $\omega_4$ and $\omega_1$ intersect at $D_1$ and $D_2$. Prove that if $A_1$, $B_1$, $C_1$ and $D_1$ are collinear or concyclic, then so are $A_2$, $B_2$, $C_2$ and $D_2$.

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Solution to Problem 6
Invert with respect to $A_1$ (above right). Then $A_1B_1C_1D_1$, $\omega_1$ and $\omega_2$ become the sides of triangle $A_2'B_2'C_2$. Since $B_2'B_1'C_1'C_2$ is cyclic, $\angle A_2'B_2'C_2 = \angle C_1'C_2'B_1$. Similarly, $\angle A_2'D_2'C_2' = \angle C_1'C_2'D_1'$. Since $A_1$, $B_1$, $C_1$ and $D_1$ are either collinear or concyclic, $C_1'$ lies on $B_1'D_1'$. Hence $\angle C_2'C_1'B_1' + \angle C_2'C_1'D_1' = 180^\circ$. It follows that $\angle A_2'B_2'C_2' + \angle A_2'D_2'C_2' = 180^\circ$, so that $A_2'$, $B_2'$, $C_2'$ and $D_2'$ are concyclic. Hence $A_2'$, $B_2'$, $C_2'$ and $D_2'$ are either collinear or concyclic.

Problem 7 (below left)
$A$, $B$ and $C$ are three points on a line and $P$ is a point not on this line. Prove that the circumcentres of triangles $PAB$, $PBC$ and $PCA$ are concyclic with $P$. 

Solution to Problem 7
Let $F$, $D$ and $E$ be the respective circumcentres. Invert with respect to $P$ (above right). Then the circles become the sides of triangle $A'B'C'$. The images $D'$, $E'$ and $F'$ of the circumcentres are the reflections of $P$ across the respective sides.

Hence the midpoints $K'$, $L'$ and $M'$ of $PD'$, $PE'$ and $PF'$ are the feet of perpendiculars from $P$ to the sides of triangle $A'B'C'$. Since $A$, $B$ and $C$ are collinear, $P$ lies on the circumcircle of triangle $A'B'C'$. It follows that $M'K'L'$ is the Simson line of triangle $A'B'C'$, so that $F'$, $D'$ and $E'$ are also collinear. Hence the circumcentres are concyclic with $P$. 

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Problem 8
Prove Ptolemy’s Inequality which states that \( AB \cdot CD + AD \cdot BC \geq AC \cdot BD \) for any convex quadrilateral \( ABCD \), with equality if and only if the quadrilateral is cyclic. (Hint: Because this is quantitative, expect to use the “polar-coordinate” definition of inversion.)

Solution to Problem 8
Invert with respect to \( A \):

Now \( B'D' = \frac{BD \cdot r^2}{AB \cdot AD} \), where \( r \) is the radius of inversion. Similarly,

\[ B'C' = \frac{BC \cdot r^2}{AB \cdot AC} \quad \text{and} \quad C'D' = \frac{CD \cdot r^2}{AC \cdot AD}. \]

By the Triangle Inequality, \( B'D' \leq B'C' + C'D' \). Substituting in this the above expressions, we have

\[ \frac{BD}{AB \cdot AD} \leq \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD}, \]

or \( AC \cdot BD \leq BC \cdot AD + CD \cdot AB \). Equality holds if and only if \( C' \) is collinear with \( B' \) and \( D' \). Since the circumcircle of triangle \( BAD \) turns into the line \( B'D' \), this holds if and only if \( C \) is concyclic with \( A, B \) and \( D \).

Problem 9 (below)
Prove that the circle which passes through the midpoints of the sides of a triangle is tangent to the triangle’s incircle and excircles.
Solution to Problem 9
We shall prove that the midpoint circle is tangent to the incircle and the excircle facing $B$. By symmetry, it will be tangent to the other two excircles. Let $BC$, $CA$ and $AB$ be tangent to the excircle facing $B$ at $K$, $L$ and $M$, and the incircle at $N$, $P$ and $Q$, respectively.

![Diagram showing the excircles and incircle](image)

Now

$$AB + AC - BC = (AQ + BQ) + (AN + CN) - (BP + CP) = 2AN$$

and

$$AB + AC - BC = (BK - AK) + (AM + CM) - (BL - CL) = 2CM.$$ 

Hence $AN = CM$. Since $E$ is the midpoint of $AC$, it is also the midpoint of $NM$.

Invert with respective to $E$ and choose $EM = EN = \frac{BC - AB}{2}$ as the radius of inversion. Then both circles are orthogonal to the circle of inversion, and coincide with their respective images.

Let $XY$ be the other common interior tangent of these two circles, with $X$ on $AB$ and $Y$ on $BC$. Let $XY$ intersect $DE$ at $D'$ and $EF$ at $F'$. If we can prove that $D'$ and $F'$ are the images of $D$ and $F$ respectively, then the midpoint circle inverts into the line $XY$, and the desired result follows. By symmetry, we have $BX = BC$ and $BY = BA$. Note that $XFF'$ and $XBY$ are similar triangles.

It follows that

$$FF' = \frac{BY \cdot XF}{BX} = \frac{BA(BX - BF)}{BC} = \frac{BA(2BC - BA)}{2BC},$$

$$EF' = EF - FF' = \frac{BC}{2} - \frac{2BC \cdot BA - BA^2}{2BC} = \frac{(BC - BA)^2}{2BC},$$

$$EF \cdot EF' = \frac{BC}{2} \cdot \frac{(BC - BA)^2}{2BC} = \frac{(BC - BA)^2}{4}. $$

Hence $F'$ is indeed the inversive image of $F$. From the similar triangles $XBY$ and $D'DY$, we can deduce in an analogous manner that $D'$ is in fact the inversive image of $D$. 

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Problem 10 (below left)
From a point $O$ are four rays $OA$, $OC$, $OB$ and $OD$ in that order, such that $\angle AOB = \angle COD$. A circle tangent to $OA$ and $OB$ intersects a circle tangent to $OC$ and $OD$ at $E$ and $F$. Prove that $\angle AOE = \angle DOF$.

Solution to Problem 10.
Let $\omega_1$ be the circle tangent to $OA$ at $A$ and to $OB$ at $B$. Let $\omega_2$ be the circle tangent to $OC$ at $C$ and to $OD$ at $D$. If $\omega_1$ and $\omega_2$ are of the same size, then both $E$ and $F$ will lie on the bisector of $\angle COB$, and the desired result follows immediately. Thus we may assume that $\omega_1$ is larger than $\omega_2$.

Invert with respect to $O$ so that $A$ and $B$ coincide with their respective images $A'$ and $B'$ (above right). The rays $OA$, $OC$, $OB$ and $OD$ become the rays $OA'$, $OC'$, $OB'$ and $OD'$ respectively. The circle $\omega_1$ coincides with its image $\omega_1'$ while the image of the circle $\omega_2$ is another circle $\omega_2'$. Note that the image $E'$ of $E$ is collinear with $E$ and $O$, and the image $F'$ of $F$ is collinear with $F$ and $O$.

We now go from the first diagram to the second in a different way. First we perform a reflection about the bisector of $\angle COB$, and then a dilation from $O$ so that $D$ is mapped into $A'$. Note that the rays $OA$, $OC$, $OB$ and $OD$ become the rays $OD'$, $OB'$, $OC'$ and $OA'$ respectively, while the circle $\omega_2$ becomes the circle $\omega_1'$. By inversion, $OD \cdot OD' = OA^2$. Since $D$ is mapped into $A'$, $A$ is mapped into $D'$ so that the circle $\omega_1$ becomes $\omega_2'$. It follows that $F$ is mapped into $E'$ while $E$ is mapped into $F'$. Thus the rays $OE$ and $OF$ become each other, and the desired result follows.