Application of Inversive Methods to Euclidean Geometry

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Place a sphere on top of the Euclidean plane so that its south pole $S$ is at the origin. Let $N$ be the north pole. For any point $Q \neq N$ on the sphere, the point $P$ of intersection of the extension of $NQ$ with the plane is called its image under the stereographic projection from $N$.

Of course, it would be tidier if $N$ had an image as well. How would it behave? As $Q$ approaches $N$ from any direction, the projection $P$ “approaches infinity” in the sense of becoming arbitrarily far away from $S$. If we add a point at infinity $I$ to the Euclidean plane, with the property that a sequence $(P_j)$ is defined to converge to $I$ if and only if $|P_j|$ increases without bound, we will have what is known as the inversive plane. Think of the sphere as a balloon and the point $N$ as a puncture. If we stretch the balloon out onto the plane, we can see that the point $I$ is in every direction!

We define the point $I$ to be the projection of $N$. It is called the ideal point, and lies on every straight line. To see this, consider a straight line $\ell$ on the inversive plane and the plane passing through $N$ and $\ell$. The cross-section with the sphere is a circle passing through the point $N$, justifying the statement that $I$ lies on every straight line. In fact, it closes the straight line into something like a circle.

Inversion

For any circle $\Sigma$ with center $O$ and radius $R$, and any point $A \neq O,I$, we define the inverse point of $A$ with respect to $\Sigma$ to be the point on the ray $\overrightarrow{OA}$ at distance $R^2/|OA|$ from $O$. This is readily seen to be an involution (self-inverse map). The points $O$ and $I$ are defined to invert into each other. We consider straight lines to be “circles passing through $I$”. Inversion in a straight line is defined to be reflection: the point $I$ is fixed under reflections. The geometry resulting from (and preserved by) these mappings is called inversive geometry. For a full introduction to inversive geometry, the reader is referred to any good undergraduate geometry textbook, such as Pedoe [1] (chapter VI) or Baragar [2] (chapter 7).

Exercise 1 Inversion fixes exactly the points of $\Sigma$. It maps points inside $\Sigma$ to points outside $\Sigma$ and vice versa.

The next result is a very useful lemma. Note the order in which the points of the triangles are specified - this is important!

Exercise 2 Let $P, Q, and the center of inversion $O$ not be collinear, and let $P, Q$ invert to $P', Q'$. Then the triangles $\triangle OPQ$ and $\triangle OQ'P'$ are similar.

The reflection of a circle in a line is always a circle. Something similar is true for inversions.

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Exercise 3  Inversion maps circles not passing through $O$ to circles, circles passing through $O$ to straight lines not through $O$, straight lines passing through $O$ to themselves, and other straight lines to circles.

We can define the angle between two circles, or between a circle and a line, at a point $P$ to be the angle between the tangent lines. Reflection preserves these, of course — so does inversion.

Exercise 4  Show that inversion preserves angles, whether between two lines, a line and a circle, or two circles.

Reflection in a line $L$ maps any line or circle that is orthogonal to $L$ to itself. (Note that if a circle meets a line or another circle twice, it makes the same angle at each intersection point. Thus “orthogonal” is well defined here and in the following exercise.)

Exercise 5  Show that inversion maps any circle orthogonal to $\Sigma$ to itself.

Exercise 6  If a circle $C$ cuts $\Sigma$, so does its inverse. If a circle $C$ is tangent to $\Sigma$, so is its inverse. If a circle $C$ contains $O$ in its interior, so does its inverse.

Exercise 7  The Euclidean construction for an inverse point is simple enough to find by trial and error.

(i) Given $O$ and $\Sigma$, and a point $P$ inside $\Sigma$, construct the inverse point $P'$.

(ii) Given $O$ and $\Sigma$, and a point $P$ outside $\Sigma$, construct the inverse point $P'$.

Reflection preserves reflections: that is, a mirror seen in a mirror acts like a mirror. Something similar holds for inversions:

Exercise 8  If $P$ and $P'$ are inverse with respect to $C$, and their inverses with respect to $\Sigma$ are $\overline{P}, \overline{P}'$, and $\overline{C}$ respectively, then $\overline{P}$ and $\overline{P}'$ are inverses with respect to $\overline{C}$.

We might wonder if inversion preserves circle centers, but it doesn’t. (It’s easy to find a counterexample — find one!) There is a way to find the center of an inverse circle, though.

Exercise 9  If $C$ and $\overline{C}$ are inverses with respect to $\Sigma$, then the center $A$ of $\overline{C}$ is found as follows. Let $B$ be the inverse of $O$ in $C$; then $A$ is the inverse of $B$ in $\Sigma$.

Problems

Problem 1 (below left)  Three circles $\omega_1$, $\omega_2$ and $\omega_3$ pass through $O$. $C$ is the other point of intersection of $\omega_1$ and $\omega_2$, $A$ is the other point of intersection of $\omega_2$ and $\omega_3$, and $B$ is the other point of intersection of $\omega_3$ and $\omega_1$. The extension of $AO$ intersects $\omega_1$ again at $D$, the extension of $BO$ intersects $\omega_2$ again at $E$, and the extension of $CO$ intersects $\omega_3$ again at $F$. Prove that if $OE$ and $OF$ are diameters of $\omega_2$ and $\omega_3$ respectively, then $OD$ is a diameter of $\omega_1$.
Problem 2 (above right)

Two circles $\omega_1$ and $\omega_2$ are tangent externally to each other at $A$. A common exterior tangent touches $\omega_1$ at $P$ and $\omega_2$ at $Q$. The other common exterior tangent touches $\omega_1$ at $R$ and $\omega_2$ at $S$. Prove that the circumcircles of triangles $PAQ$ and $RAS$ are tangent to each other.

Problem 3 (below left)

$AB$, $AC$ and $AD$ are three chords on a circle. Circles with $AB$ and $AC$ as diameters intersect at $E$, circles with $AB$ and $AD$ as diameters intersect at $F$, and circles with diameters $AC$ and $AD$ intersect at $G$. Prove that $E$, $F$ and $G$ are collinear.

Problem 4 (above right)

Three circles $\omega_1$, $\omega_2$ and $\omega_3$ pass through $O$. $B$ is the other point of intersection of $\omega_1$ and $\omega_2$, $C$ is the other point of intersection of $\omega_2$ and $\omega_3$, and $A$ is the other point of intersection of $\omega_3$ and $\omega_1$. The tangent to $\omega_2$ at $O$ intersects $BC$ at $D$, the tangent at $O$ to $\omega_3$ intersects $CA$ at $E$, and the tangent at $O$ to $\omega_1$ intersects $AB$ at $F$. Prove that $D$, $E$ and $F$ are collinear.

Problem 5 (below left)

Four circles $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ are such that $\omega_1$ and $\omega_2$ touch at $A$, $\omega_2$ and $\omega_3$ touch at $B$, $\omega_3$ and $\omega_4$ touch at $C$ and $\omega_4$ and $\omega_1$ touch at $D$. Prove that $A$, $B$, $C$ and $D$ are concyclic.

Problem 6 (above right)
Four circles $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ are such that $\omega_1$ and $\omega_2$ intersect at $A_1$ and $A_2$, $\omega_2$ and $\omega_3$ intersect at $B_1$ and $B_2$, $\omega_3$ and $\omega_4$ intersect at $C_1$ and $C_2$, and $\omega_4$ and $\omega_1$ intersect at $D_1$ and $D_2$. Prove that if $A_1$, $B_1$, $C_1$ and $D_1$ are collinear or concyclic, then so are $A_2$, $B_2$, $C_2$ and $D_2$.

Problem 7 (below)
$A$, $B$ and $C$ are three points on a line and $P$ is a point not on this line. Prove that the circumcentres of triangles $PAB$, $PBC$ and $PCA$ are concyclic with $P$.

Problem 8
Prove Ptolemy’s Inequality which states that $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ for any convex quadrilateral $ABCD$, with equality if and only if the quadrilateral is cyclic. (Hint: Because this is quantitative, expect to use the “polar-coordinate” definition of inversion.)

Problem 9 (below left)
Prove that the circle which passes through the midpoints of the sides of a triangle
is tangent to the triangle’s incircle and excircles.

**Problem 10** (above right)
From a point $O$ are four rays $OA$, $OC$, $OB$ and $OD$ in that order, such that $\angle AOB = \angle COD$. A circle tangent to $OA$ and $OB$ intersects a circle tangent to $OC$ and $OD$ at $E$ and $F$. Prove that $\angle AOE = \angle DOF$.

The solution to Problem 1 is given as an example. We leave the others to the reader!

**Solution (to Problem 1)** Invert with respect to any circle with center $O$. Then the three circles turn into triangle $A'B'C'$ while the radial lines $OA$, $OB$ and $OC$ invert to themselves. That $OE$ is a diameter of $\omega_2$ means that $B'E'$ is orthogonal to $A'C'$. Similarly, $C'F'$ is orthogonal to $A'B'$. Hence $O$ is the orthocentre of triangle $A'B'C'$, so that $A'O$ is orthogonal to $B'C'$. It follows that $OD$ is indeed a diameter of $\omega_1$.

**References**