No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


An asterisk (*) after a number indicates that a problem was proposed without a solution.


Let $n$ be a positive integer, and let

$$a(n) = \left| \sum_{j=0}^{3n} (-2)^j \left( \binom{6n + 2 - j}{j + 1} + \binom{6n + 1 - j}{j} \right) \right|.$$

Prove that

(a) $a(n) = 3$ if and only if $n = 1$, and

(b) the sequence $\{a(n)\}_{n=1}^{\infty}$ is strictly increasing.

We give the solution to part (b) by C. R. Pranesachar.

We shall prove that the statement is false. In fact, there exist infinitely many positive integers $n$ such that $a(n) > a(n + 1)$.

Let

$$b(n) = \sum_{j \geq 1} (-2)^j \left( \binom{n + 1 - j}{j + 1} + \binom{n - j}{j} \right), \quad n \geq 0.$$  

Also let $c(n) = b(6n + 1)$. Then $a(n) = |c(n)|$, as the full range of $j$ is used. Since $\sum_{j \geq 0} \binom{n-j}{j}$ is the coefficient of $x^n$ in the series

$$1 + x(1 + x) + x^2(1 + x)^2 + \cdots + x^n(1 + x)^n + \cdots = \frac{1}{1-x-x^2},$$

we see that $\sum_{j \geq 0} (-2)^j \binom{n-j}{j}$ is the coefficient of $x^n$ in the series

$$1 + x(1 - 2x) + x^2(1 - 2x)^2 + \cdots + x^n(1 - 2x)^n + \cdots = \frac{1}{1-x+2x^2}.$$  

Now,

$$\sum_{j \geq 0} (-2)^j \binom{n+1-j}{j+1} = \sum_{j \geq 1} (-2)^{j-1} \binom{n+2-j}{j} = \frac{1}{2} - \frac{1}{2} \sum_{j \geq 0} (-2)^j \binom{n+2-j}{j},$$

so this sum is the coefficient of $x^{n+2}$ in

$$\frac{1}{2(1-x)} - \frac{1}{2(1-x+2x^2)}.$$
Thus $b(n)$ is the coefficient of $x^{n+2}$ in
\[
\frac{x^2}{1 - x + 2x^2} + \frac{1}{2(1 - x)} - \frac{1}{2(1 - x + 2x^2)} = \frac{1}{2(1 - x)} + \frac{2x^2 - 1}{2(1 - x + 2x^2)} - \frac{x - 2}{2(1 - x + 2x^2)}.
\]
Hence $b(n) = \frac{1}{2} + A_0 \alpha^n + B_0 \beta^n$, where $\alpha$ and $\beta$ are the roots of $\lambda^2 - \lambda + 2 = 0$, and $A_0, B_0$ are two fixed numbers. We have
\[
\alpha = \frac{1 + i\sqrt{7}}{2}, \quad \beta = \frac{1 - i\sqrt{7}}{2}.
\]
Using the initial values $b(0) = 2$ and $b(1) = 3$, we get easily that
\[
A_0 = \frac{3 - i\sqrt{7}}{4}, \quad B_0 = \frac{3 + i\sqrt{7}}{4}.
\]
Since $a(n) = |c(n)|$, we compute $c(n)$. In fact
\[
c(n) = b(6n + 1) = \frac{1}{2} + \left(\frac{3 - i\sqrt{7}}{4}\right)\alpha^{6n+1} + \left(\frac{3 + i\sqrt{7}}{4}\right)\beta^{6n+1}
\]
\[
= \frac{1}{2} + \left(\frac{5 + i\sqrt{7}}{4}\right)\alpha^{6n} + \left(\frac{5 - i\sqrt{7}}{4}\right)\beta^{6n}.
\]
Since
\[
\alpha^6 = \left(\frac{1 + i\sqrt{7}}{2}\right)^6 = \frac{9 + i\sqrt{7}}{2} \quad \text{and} \quad \beta^6 = \frac{9 - i\sqrt{7}}{2},
\]
we have
\[
c(n) = \frac{1}{2} + \left(\frac{5 + i\sqrt{7}}{4}\right)\left(\frac{9 + i\sqrt{7}}{2}\right)^n + \left(\frac{5 - i\sqrt{7}}{4}\right)\left(\frac{9 - i\sqrt{7}}{2}\right)^n, \quad n \geq 1.
\]
If we set
\[
A = \frac{5 + i\sqrt{7}}{4}, \quad B = \frac{5 - i\sqrt{7}}{4},
\]
then
\[
A^2 = \frac{9 + 5i\sqrt{7}}{8}, \quad B^2 = \frac{9 - 5i\sqrt{7}}{8}.
\]
Hence
\[
b(6n + 1) = \frac{1}{2} + A^2(4A^2)^n + B(4B^2)^n = \frac{1}{2} + 4^n(A^{2n+1} + B^{2n+1}).
\]
Now we may take $A = \sqrt{2}e^{i\theta}, B = \sqrt{2}e^{-i\theta}$, where $\theta = \arccos\left(\frac{5}{4\sqrt{7}}\right)$ is acute. So
\[
c(n) = \frac{1}{2} + 8^n(2\sqrt{2})\cos(2n + 1)\theta.
\]
Hence
\[ a(n) = |c(n)| = \left| \frac{1}{2} + 8^n (2\sqrt{2}) \cos(2n + 1) \theta \right|, \quad n \geq 1. \]

Now we exploit the properties of \( \theta \). Surprisingly, \( \theta \) has some nice ‘solution-friendly’ properties that are precisely needed. Firstly,
\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 2 \left( \frac{25}{32} \right) - 1 = \frac{9}{16}. \]

From this we infer that \( \theta \) cannot be a rational multiple of \( \pi \) (the proof is left as an exercise for the reader). Therefore, the set \( \{ \cos(2n + 1) \theta : n \in \mathbb{N} \} \) is dense in \([-1, 1]\). The next property of \( \theta \) that we use is as follows: since \( \frac{5}{4\sqrt{2}} > \frac{3}{2} \), we have \( \cos \theta > \cos 30^\circ \), and so \( \theta < 30^\circ \). Hence, if \( \phi \in (\theta, 30^\circ) \), an interval of positive length, we may write \( \phi = \theta + \tau \) for some suitable \( \tau \in (0, 30^\circ - \theta) \).

Further
\[ \cos \phi - 8 \cos(\phi + 2\theta) = \cos(\theta + \tau) - 8 \cos(3\theta + \tau) \]
\[ = \cos \tau \left( \cos \theta - 8 \sin 3\theta \right) + \sin \tau \left( 8 \sin 3\theta - \sin \theta \right) \]
\[ = \sin \tau \left( 8 \sin 3\theta - \sin \theta \right) > 0. \]

(In fact, one has \( \sin 3\theta > \sin \theta \) since \( 0 < \theta < 3\theta < 90^\circ \).)

Thus \( \cos \phi > 8 \cos(\phi + 3\theta) \). Now the set \( \{ \cos(2n + 1) \theta : n \in \mathbb{N} \} \) being dense in the interval \( (\cos 30^\circ, \cos \theta) \), we have that for some \( m \in \mathbb{N} \), we get \( (2m + 1) \theta = 2k\pi + \phi \), where \( k \in \mathbb{N} \) and \( \phi \in (\theta, 30^\circ) \). Hence
\[ \cos(2m + 1) \theta = \cos \phi > 8 \cos(\phi + 2\theta) = 8 \cos(2m + 3) \theta > 0. \]

This is sufficient to infer that \( c(m) > c(m + 1) > 0 \), and hence \( a(m) > a(m + 1) \). Thus \( a(n) > a(n + 1) \), for infinitely many \( n \) as \( (2n + 1) \theta \) visits \( (\theta, 30^\circ) \) (mod \( 2\pi \)) infinitely often. Also one has \( \theta = 27^\circ 55' 8'' \) (approximately) and so we can as well extend the interval \( (\theta, 30^\circ) \) to \( (\theta, 34^\circ) \) safely, as \( 34^\circ + 2\theta \) is still less than \( 90^\circ \).

Note that we have proved the result only for positive values of \( c(n) \). It may happen that \( |c(n)| > |c(n + 1)| \) for some negative values of \( c(n) \) also. Values of \( n \) less than 100 for which this happens are given below. These values can be obtained by giving the above numerical value for \( \theta \) and relevant values of \( n \).

\[
\begin{align*}
    c(13) &= 1305410163123, & c(14) &= 286249224103; \\
    c(42) &\approx -2.078035580 \cdot 10^{38}, & c(43) &\approx -1.327909464 \cdot 10^{38}; \\
    c(55) &\approx -1.079614797 \cdot 10^{50}, & c(56) &\approx 1.974401435 \cdot 10^{49}; \\
    c(84) &\approx 1.723273540 \cdot 10^{76}, & c(85) &\approx 4.481080748 \cdot 10^{75}; \\
    c(97) &\approx 8.905486681 \cdot 10^{87}, & c(98) &\approx -5.427940879 \cdot 10^{87}.
\end{align*}
\]

Although (b) is false, (a) may be still true and it is believed by this solver that (a) is in fact true.
Editor’s comments. The approach above might provide a solution to part (a). Starting with the expression for \(c(n)\) derived above, express everything as a rational expression in \(\alpha\): writing \(x = \alpha^{6n+3}\), manipulate \(c(n) = \pm 3\) into monic quadratic expressions and get a value of \(x\) as a complex number. A simple recurrence for the real part of powers of \(\alpha\), in reduced form, should then yield the desired result. The reader should work out the details to see if there is an unforeseen pitfall.

3911. Proposed by Paul Bracken.

Let \(x_0 \in (0, 1 - 1/a]\), where \(a > 1\), and define the sequence \(x_n = x_{n-1} - x_{n-1}^2\) for \(n \in \mathbb{N}\). Prove that \(x_n\) satisfies the inequalities

\[
\frac{x_0}{anx_0 + 1} < x_n < \frac{x_0}{nx_0 + 1}, \quad n \in \mathbb{N}.
\]

We have received eight correct solutions. We present the solution by Arkady Alt slightly modified by the editor.

Note first that since \(x_0 \in (0, 1)\) and \(x_1 - x_0 = -x_0^2 < 0\), we have \(x_1 < x_0\). Furthermore, since \(x_0 > x_0^2\), we have that \(x_1 = x_0 - x_0^2 > 0\). Hence, \(0 < x_1 < 1\).

By similar argument and induction, it is easily shown that the sequence \((x_n)\) is strictly decreasing and \(x_n \in (0, 1)\) for all \(n \in \mathbb{N}\).

Since

\[
1 = \frac{1}{x_k} = \frac{1}{x_{k-1} - x_{k-1}^2} = \frac{1}{x_k(1 - x_{k-1})} = \frac{1}{x_{k-1}} + \frac{1}{1 - x_{k-1}}.
\]

we have

\[
\frac{1}{x_k} - \frac{1}{x_{k-1}} = \frac{1}{1 - x_{k-1}}
\]

for all \(k \in \mathbb{N}\). Hence for all \(n \in \mathbb{N}\) we have

\[
\frac{1}{x_n} - \frac{1}{x_0} = \sum_{k=1}^{n} \left( \frac{1}{x_k} - \frac{1}{x_{k-1}} \right) = \sum_{k=1}^{n} \frac{1}{1 - x_{k-1}} \leq \sum_{k=1}^{n} \frac{1}{1 - x_0} = \frac{n}{1 - x_0} < \frac{n}{1 - (1 - \frac{1}{a})} = an,
\]

from which we get

\[
\frac{1}{x_n} < \frac{1}{x_0} + an = \frac{anx_0 + 1}{x_0},
\]

so

\[
\frac{x_0}{anx_0 + 1} < x_n. \quad (1)
\]

Using (1), we get

\[
\frac{1}{x_n} - \frac{1}{x_0} = \sum_{k=1}^{n} \frac{1}{1 - x_{k-1}} \geq \sum_{k=1}^{n} \frac{1}{1 - x_n} > \sum_{k=1}^{n} \frac{1}{1 - \frac{x_n}{anx_0 + 1}} = \frac{n}{1 - \frac{x_n}{anx_0 + 1}} = \frac{n(anx_0 + 1)}{anx_0 + 1 - x_0} > n.
\]

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so
\[
\frac{1}{x_n} > \frac{1}{x_0} + n = \frac{n x_0 + 1}{x_0}.
\]
Hence,
\[
x_n < \frac{x_0}{n x_0 + 1}.
\] (2)
From (1) and (2), the proof is complete.

3912. Proposed by Michel Bataille.

Let \( ABC \) be a scalene triangle with no right angle and \( H \) as its orthocenter. If \( A_1, B_1 \) and \( C_1 \) are the midpoints of \( BC, CA \) and \( AB \) respectively, prove that the orthocenters of \( HAA_1, HBB_1 \) and \( HCC_1 \) are collinear.

We received seven submissions, five of which were correct and two incomplete. We present a composite of the solutions by Šefket Arslanagić and by the proposer.

Define the point \( A' \) to be the reflection of \( H \) in \( A_1 \). Then \( HBA'C \) is a parallelogram (because its diagonals bisect one another), which implies that \( A'C \perp AC \) (because \( BH \) is parallel to \( A'C \) and perpendicular to \( AC \)). Similarly, \( A'B || CH \) so that \( A'B \perp AB \); consequently, \( AA' \) is a diameter of the circumcircle \( \Gamma \) of \( \triangle ABC \).

Let \( K \) be the orthogonal projection of \( A \) onto the line \( A_1H \) (note that \( K \neq A \) because \( \angle BAC \neq 90^\circ \)). This point \( K \) is on the circle \( \gamma_1 \) with diameter \( AA_1 \) and, from the preceding remark (which implies that \( A', A_1, K, H \) are collinear), is also on \( \Gamma \). It follows that \( AK \) is the radical axis of the circles \( \Gamma \) and \( \gamma_1 \).

Let \( \gamma \) denote the Euler (or nine-point) circle of \( \triangle ABC \) (which passes through the feet of the altitudes and the midpoints of the sides). Since \( \angle BAC \neq 90^\circ \), we have \( \gamma \neq \gamma_1 \). In addition, both \( \gamma_1 \) and \( \gamma \) pass through \( A_1 \) and the orthogonal projection \( D \) of \( A \) onto \( BC \) (which are distinct points because \( AB \neq AC \)). Thus, the line \( BC \) is the radical axis of \( \gamma_1 \) and \( \gamma \). Now, the orthocentre \( P \) of \( \triangle AH H_1 \), which is the point of intersection of \( AK \) and \( BC \), is the radical center of the
circles $\Gamma, \gamma, \gamma_1$. Thus, $P$ is on the radical axis of the circles $\Gamma$ and $\gamma$. The same is true of the orthocentres of triangles $HBB_1$ and $HCC_1$. The desired result follows immediately.

But we can deduce yet more: these three orthocentres lie on the orthic axis of $\triangle ABC$ (which, consequently, must coincide with the radical axis of $\Gamma, \gamma$), as we now show. If $E$ and $F$ are the feet of the altitudes from the vertices $B$ and $C$ of $\triangle ABC$, then they lie on the circle whose diameter is $AH$, and that circle, call it $\gamma_2$, also contains $K$ (because $AK \perp HK$). Thus the radical axis of $\gamma_2$ and $\gamma_1$ must be $AK$, while the radical axis of $\gamma_2$ and the Euler circle $\gamma$ is $EF$. Putting these lines together with the radical axis $BC$ of circles $\gamma$ and $\gamma_1$, we see that the radical centre of these three circles must be the common point of $AK, BC$, and $EF$, which we know to be $P$ (the intersection of $BC$ and $AK$). By analogous arguments, the orthocentres of triangles $HBB_1$ and $HCC_1$ must likewise be the intersections of the sides $DE$ and $DF$ of the orthic triangle $DEF$ with the corresponding sides $AB$ and $AC$ of the initial triangle. Of course, a triangle and its orthic triangle are perspective from the orthocentre, whence they must be perspective from a line, namely the orthic axis. We have just seen that the corresponding sides of the two triangles intersect in the orthocentres of $HAA_1, HBB_1, HCC_1$, which completes a second proof that these three points are collinear.

Editor’s comments. Both incomplete submissions provided neat arguments to show that the three orthocentres satisfy one of the conditions required for the converse of Menelaus’s theorem, but (as was pointed out in the editorial comments following problem 3885 [2014 : 399]) a second condition must be satisfied: zero or two of the orthocentres must lie on the sides of $\triangle ABC$ (while one or three lie on the extensions of those sides).

3913. Proposed by Ovidiu Furdui.

Calculate

$$\int_0^\infty \int_0^\infty \frac{dx dy}{(e^x + e^y)^2}.$$

We received ten correct solutions and two incorrect submissions. We present the solution by Madhav Modak.

Denote the repeated integral by $I$. Then by change of variables we have

$$I = \int_0^\infty \int_0^\infty \frac{e^{-2y}e^{-2x}}{(e^{-x} + e^{-y})^2} dx dy$$

$$= \int_0^\infty \int_{1+e^{-y}}^{e^{-y}} \frac{-e^{-2y}}{t^2} (t - e^{-y}) dt dy$$

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where \( e^{-x} + e^{-y} = t \), and \(-e^{-x}dx = dt\). Proceeding with the integral, we have

\[
I = \int_0^\infty \int_0^{1+e^{-y}} e^{-2y} \left( \frac{1 - e^{-y}}{t^2} \right) dt \, dy
\]

\[
= \int_0^\infty e^{-2y} \left[ \log t + \frac{e^{-y}}{t} \right]_{e^{-y}}^{1+e^{-y}} dy
\]

\[
= \int_0^\infty e^{-2y} \left[ \log t + e^{-y} \left( \frac{1}{1+e^{-y}} - \frac{1}{e^{-y}} \right) \right] dy
\]

\[
= \int_0^\infty e^{-2y} \left[ \log(1+e^{-y}) + e^{-y} \right] dy + \int_0^\infty \left( \frac{e^{-3y}}{1+e^{-y}} - e^{-2y} \right) dy
\]

\[
= \left[ \frac{e^{-2y}}{-2} \log(1+e^{-y}) \right]_0^\infty - \int_0^\infty \frac{e^{-2y}}{-2} \cdot \frac{-e^{-y}}{1+e^{-y}} dy + \int_0^\infty ye^{-2y} dy
\]

\[
= \int_0^\infty \left( \frac{e^{-3y}}{1+e^{-y}} - e^{-2y} \right) dy
\]

\[
= \frac{1}{2} \log 2 - \frac{1}{2} \int_0^\infty \frac{e^{-3y}}{1+e^{-y}} dy + \left[ \frac{e^{-2y}}{-2} \right]_0^\infty - \int_0^\infty \frac{e^{-2y}}{-2} dy
\]

\[
= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^\infty \frac{e^{-3y}}{1+e^{-y}} dy - \frac{1}{2} \int_0^\infty e^{-2y} dy.
\]

Letting \( 1 + e^{-y} = w \), and \(-e^{-y}dy = dw\), we have

\[
= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{-(w-1)^2}{w} dw - \frac{1}{2} \left[ \frac{e^{-2y}}{-2} \right]_0^\infty
\]

\[
= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 (w-2 + \frac{1}{w}) dw - \frac{1}{4}
\]

\[
= \frac{1}{2} \log 2 + \frac{1}{2} \left( \frac{3}{2} - 2 + \log 2 \right) - \frac{1}{4},
\]

so that

\[
I = \log 2 - \frac{1}{2}.
\]

3914. \textit{Proposed by George Apostolopoulos; generalized by the Editorial Board.}

Let \( ABC \) be a triangle with circumradius \( R \), inradius \( r \) and semiperimeter \( s \), such that \( s = kr \). Prove that

\[
\frac{2k}{3\sqrt{3}} < \frac{R}{r} < \frac{k^2 - 3}{12}.
\]

We received 13 correct solutions. \textit{We present a hybrid of several solutions that efficiently applied formulae (implicitly and explicitly) from O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.}
As noted in most of the solutions, we assume that the triangle is not equilateral, as both of the inequalities become equations if the given triangle is equilateral.

Using $k = s/r$, the left-hand inequality is equivalent to $2s < 3R\sqrt{3}$, which is inequality 5.3 from Bottema et al.

The right-hand inequality is equivalent to

$$3r(4R + r) < s^2,$$

which is given both in 5.5 and 5.6 of Bottema et al.

3915. Proposed by Marcel Ciiriță.

Let $M$ and $N$ be points on the sides $AB$ and $AC$, respectively, of triangle $ABC$, and define $O = BN \cap CM$. Show that there are infinitely many examples (that are not affinely equivalent) for which the areas of the four regions $MBO, BCO, CNO$ and $AMON$ are all integers.

We received four correct solutions to this problem, each utilizing a different construction. We feature three of them.

Solution 1, based on the construction by Digby Smith.

For arbitrary $p,q \in \mathbb{N}$ with $p > q$, the numbers $p^2 - q^2$, $2pq$, and $p^2 + q^2$ (and any multiples thereof) form a Pythagorean Triple. The configuration $AMBCN$ from the problem is defined by

$$BC = 4pq(p^2 - q^2)(p^2 + q^2),$$
$$BM = CN = 4pq(p^2 - q^2)(2pq),$$
$$BN = CM = 4pq(p^2 - q^2)(p^2 - q^2).$$

Then $ABC$ is isosceles and $BMC$ and $CNB$ are congruent and right angled. By their definitions, the areas of both $BMC$ and $CNB$ are equal to $BM \cdot CM/2$ and thus integers. If we define $D$ to be the midpoint of $BC$, then $BDO \sim BNC$ and

$$DO = \frac{NC \cdot BD}{BN} = \frac{8p^2q^2(p^2 - q^2) \cdot 2pq(p^2 - q^2)(p^2 + q^2)}{4pq(p^2 - q^2)^2} = 4p^2q^2(p^2 + q^2).$$

The area of $BCO$ is equal to $BD \cdot DO/2$ and therefore integer. It follows that the areas of $BMO$ and $CNO$ are also integers. Finally $ADC \sim BNC$ and thus

$$AD = \frac{BN \cdot DC}{NC} = \frac{4pq(p^2 - q^2)^2 \cdot 2pq(p^2 - q^2)(p^2 + q^2)}{8p^2q^2(p^2 - q^2)} = (p^2 - q^2)^2(p^2 + q^2).$$

The area of $ABC$ is equal to $BC \cdot AD/2$ and therefore integer. By subtracting the areas of $BMO$, $BCO$, and $CNO$, we obtain finally that the area of $AMON$ is also an integer. As $p,q$ can be arbitrarily chosen, we obtain an infinite number of configurations that are not affinely equivalent (e.g. choose all pairs $p,q$ with $\gcd(p,q) = 1$).

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Solution 2, abridged version of the solution by the Missouri State University Problem Solving Group.

Let $A = (0, 0), B = (1, 0), C = (0, 1), M = (a, 0),$ and $N = (0, b),$ where $a$ and $b$ are rational and $0 < a < b < 1.$ The equations of the lines $BN$ and $CM$ have rational coefficients, so the coordinates of $O$ are rational. The area of a triangle with vertices $(x_1, y_1), (x_2, y_2),$ and $(x_3, y_3)$ is

$$\frac{1}{2} \left| \det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \right|.$$ 

Therefore, the areas of $MBO,$ $BCO,$ $CNO,$ and $AMON$ are all rational. By stretching the triangle $ABC,$ the corresponding areas can be made to be integers. Since stretching does not alter the ratios $AM/MB$ and $AN/NC,$ the configurations are not affinely equivalent for distinct choices of $a$ and $b.$

Solution 3, by Titu Zvonaru.

Let $a, b, m, n$ be positive integers and let $ABC$ be a triangle with $BC = 2a$ and $h_A = b(m + 1)(n + 1)(m + n + 1).$ Choose the points $M$ and $N$ on $AB$ and $AC$ such that $\frac{BM}{BA} = \frac{1}{m + 1}, \frac{CN}{CA} = \frac{1}{n + 1}.$ Denote by $[XY \ldots Z]$ the area of the polygon $XY \ldots Z.$ Then

$$[BMC] = \frac{[ABC]}{m + 1}, \quad [CNB] = \frac{[ABC]}{n + 1}.$$

Suppose that $\overline{AO}$ intersects $\overline{BC}$ at $A'.$ By Van Aubel’s Theorem for Cevian triangles we obtain $\frac{AO}{OA'} = \frac{AM}{MB} + \frac{AN}{NC} = m + n$ and therefore $OA' = AA'/(m + n + 1).$ It follows that

$$[BOC] = \frac{[ABC]}{m + n + 1}.$$

Thus the areas $[ABC], [BMC], [CNB],$ and $[BOC]$ are all integers and by taking differences of these areas so are $[MBO], [CNO],$ and $[AMON].$

3916. Proposed by Nathan Soedjak.

Let $a, b, c$ be positive real numbers. Prove that

$$\left( \frac{ab}{c} \right)^2 + \left( \frac{bc}{a} \right)^2 + \left( \frac{ca}{b} \right)^2 \geq 3 \left( \frac{ab + bc + ca}{a + b + c} \right)^2.$$

There were 23 correct solutions, with two solutions from one solver, as well as a Maple verification. We present a sampling of the different approaches.
**Solution 1, by Mohammed Aassila.**

Note that $x^2 + y^2 + z^2 \geq xy + yz + zx$ and $(x + y + z)^2 \geq 3(xy + yz + zx)$ for real $x, y, z$. The left side of the inequality is not less than $b^2 + c^2 + a^2$. However

$$(a^2 + b^2 + c^2)(a + b + c)^2 \geq (ab + bc + ca)[3(ab + bc + ca)] = 3(ab + bc + ca)^2,$$

and the desired result follows.

**Solution 2, by Michel Bataille.**

By homogeneity, we may suppose that $a + b + c = 1$. The inequality is then equivalent to

$$(ab)^4 + (bc)^4 + (ca)^4 \geq 3(a^2b^2c^2)(ab + bc + ca)^2.$$

Observe that

$$x^4 + y^4 + z^4 = \frac{1}{4}[(x^4 + x^4 + y^4 + z^4) + (x^4 + y^4 + y^4 + z^4) + (x^4 + y^4 + z^4 + z^4)]$$

$$\geq x^2yz + xy^2z + xyz^2 = xyz(x + y + z),$$

and $(x + y + z)^2 \geq 3(xy + yz + zx)$. Applying these inequalities leads to

$$(ab)^4 + (bc)^4 + (ca)^4 = [(ab)^4 + (bc)^4 + (ca)^4][(a + b + c)^2]$$

$$\geq [a^2b^2c^2(ab + bc + ca)][3(ab + bc + ca)]$$

$$= 3(a^2b^2c^2)(ab + bc + ca)^2,$$

as desired.

**Solution 3, by Dionne Bailey, Elsie Campbell, and Charles Dimminnie; Angel Plaza; Cao Minh Quang; and Edmund Swylan, independently.**

Since $x^2 + y^2 + z^2 \geq xy + yz + zx$,

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{(ab)^2 + (bc)^2 + (ca)^2}{abc} \geq \frac{abc(a + b + c)}{abc} = a + b + c.$$

Using either the convexity of the function $x^2$ or the inequality of the root-mean-square and arithmetic mean, we find that

$$\left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 \geq \frac{1}{3}\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^2$$

$$\geq \frac{1}{3}(a + b + c)^2 = \frac{1}{3}(a + b + c)^4$$

$$\geq \frac{3(ab + bc + ca)^2}{3(a + b + c)^2} = 3\left(\frac{ab + bc + ca}{a + b + c}\right)^2.$$

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Solution 4, by Paolo Perfetti.

Since
\[
\frac{1}{4} \left( \frac{x^2y^2}{z^2} + \frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2} \right) \geq \frac{xy}{2}
\]
by the arithmetic-geometric means inequality, we can follow the strategy of Solution 2 to obtain
\[
\left( \frac{ab}{c} \right)^2 + \left( \frac{bc}{a} \right)^2 + \left( \frac{ca}{b} \right)^2 \geq \frac{(ab + bc + ca)^2}{3(ab + bc + ca)} \geq \frac{(ab + bc + ca)^2}{(a + b + c)^2}
\]
as desired.

Solution 5 by Henry Ricardo.

We have
\[
\left( \frac{ab}{c} \right)^2 + \left( \frac{bc}{a} \right)^2 + \left( \frac{ca}{b} \right)^2 = \frac{1}{2} \left[ a^2 \left( \frac{b^2}{c^2} + \frac{c^2}{b^2} \right) + b^2 \left( \frac{a^2}{c^2} + \frac{c^2}{a^2} \right) + c^2 \left( \frac{b^2}{a^2} + \frac{a^2}{b^2} \right) \right]
\]
\[
\geq a^2 + b^2 + c^2
\]
\[
\geq \frac{(a + b + c)^2}{3}
\]
\[
\geq 3 \left( \frac{ab + bc + ca}{a + b + c} \right)^2.
\]

3917. Proposed by Peter Y. Woo.

Given a circle \(Z\), its center \(O\), and a point \(A\) on \(Z\), with only a long unmarked ruler, and no compass, can you draw:

i) points \(B, C\) and \(D\) on \(Z\) so that \(ABCD\) is a square?

ii) the square \(AOBA'\)?

iii) the points \(B, W', W\) and \(W'\) on \(Z\) such that angles \(AOB, AOW'\), \(AOW\) and \(AOW'\) are \(90^\circ, 60^\circ, 45^\circ\) and \(30^\circ\)?

There were five correct solutions to this problem. We feature the one by the Missouri State University Problem Solving Group.

We need the following basic construction: Given three collinear points \(A, B, C\) such that \(AB = BC\) and a point \(P\) not on \(\overrightarrow{AC}\), we want to construct a line through \(P\) parallel to \(\overrightarrow{AC}\). To do this, we choose a point \(Q\) on the ray \(\overrightarrow{AP}\) such that \(P\) is between \(A\) and \(Q\). Denote the intersection of \(\overrightarrow{BQ}\) and \(\overrightarrow{CP}\) by \(R\) and denote the intersection of \(\overrightarrow{AR}\) and \(\overrightarrow{QC}\) by \(S\). We claim that \(\overrightarrow{PS}\) is the line we seek. By Ceva’s theorem,
\[
\frac{AB}{BC} \cdot \frac{CS}{SQ} \cdot \frac{QP}{PA} = 1,
\]

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but since $AB = BC$ this yields $\frac{QP}{PA} = \frac{QS}{SC}$ and therefore $\vec{PS}$ and $\vec{AC}$ are parallel.

i) The intersection of $\overrightarrow{OA}$ and $Z$ gives $C$. Choose any point $X$ on $Z$ other than $A$ or $C$ and use the basic construction above to obtain a line through $X$ parallel to line $AC$. If this line only meets $Z$ in a single point let $B = X$ and $D$ be the intersection of line $OB$ and $Z$. If the line meets $Z$ in two distinct points, $S$ and $T$, let $U$ be the intersection of $\overrightarrow{AS}$ and $\overrightarrow{CT}$. Then $B$ and $D$ are the intersections of $\overrightarrow{UO}$ with $Z$. Note that by symmetry, $\overrightarrow{UO}$ is perpendicular to $\overrightarrow{AC}$, which makes $ABCD$ a square.

ii) Using the basic construction, we draw a line through $A$ parallel to $\overrightarrow{BD}$ and a line through $B$ parallel to $\overrightarrow{AC}$. Their intersection is the point $A'$.

iii) We constructed $B$ in part i). The point where $\overrightarrow{OA}$ meets $Z$ gives $W$. Let $E$ denote the intersection of $\overrightarrow{OA'}$ and $\overrightarrow{AB}$. Using the basic construction, if we draw a line $\ell$ through $E$ parallel to $\overrightarrow{AC}$, the point of intersection of $\ell$ and $Z$ that lies between $A$ and $B$ gives $W'$ (note that $\ell$ bisects $\overrightarrow{OB}$, which gives $\sin(<AOW') = 1/2$). Similarly, a line through $E$ parallel to $\overrightarrow{BD}$ gives $W''$.

3918. Proposed by George Apostolopoulos.

Let $a, b$ and $c$ be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{(ab)^{2/3} + (bc)^{2/3} + (ac)^{2/3}} < \frac{2 + \sqrt{3}}{3}.$$ 

We received 22 correct solutions and one incorrect solution. We present the solution by Cristinel Mortici, slightly modified by the editor.

Recall the Power Mean Inequality: for $x, y, z > 0$ and $m \geq n$

$$\left(\frac{x^m + y^m + z^m}{3}\right)^{1/m} \geq \left(\frac{x^n + y^n + z^n}{3}\right)^{1/n}.$$ 

The Power Mean Inequality with $m = 1$ and $n = 2/3$ gives

$$\left(\frac{(ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}}{3}\right)^{3/2} \leq \frac{ab + bc + ca}{3} \leq \frac{a^2 + b^2 + c^2}{3} = \frac{1}{3}.$$ 

It follows that

$$\left((ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}\right)^{1/2} \leq 3^{1/6}.$$ 

Finally, the Geometric Mean-Arithmetic Mean Inequality gives us

$$3^{1/6} = (1 \cdot 1 \cdot \sqrt{3})^{1/3} < \frac{1 + 1 + \sqrt{3}}{3} = \frac{2 + \sqrt{3}}{3},$$

completing the proof.

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3919. Proposed by Michel Bataille.

Let $I$ be the incentre of triangle $ABC$. The line segment $AI$ meets the incircle at $M$ and the perpendicular to $AM$ at $M$ intersects $BI$ at $N$. If $P$ is a point on the line $AI$, prove that $PC$ is perpendicular to $AI$ if and only if $PN$ is parallel to $BM$.

We received seven correct solutions. We present a composite of the similar solutions by Šefket Arslanagić and by Peter Woo.

On the one hand,

$$PN \parallel BM \iff \Delta BIM \sim \Delta NIP \iff \frac{IM}{IB} = \frac{IP}{IN}.$$ 

On the other hand,

$$PC \perp AI \iff \Delta CPI \text{ has a right angle at } P \iff \cos \angle PIC = \frac{IP}{IC}.$$ 

Let $D$ be the foot of the perpendicular from $I$ to $BC$. Then $ID = IM = r$ (the inradius), and in right triangle $BDI$ we have $IB = \frac{r}{\sin \frac{B}{2}}$, whence

$$\frac{IM}{IB} = \frac{r}{\left(\frac{r}{\sin \frac{B}{2}}\right)} = \sin \frac{B}{2}. \quad (1)$$

In right triangle $IDC$, we have

$$IC = \frac{r}{\sin \frac{C}{2}}. \quad (2)$$

Because $\angle NIM$ is exterior to $\Delta BIA$, $\angle NIM = \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}$; consequently, in right triangle $NMI$ we have

$$\cos \angle NIM = \sin \frac{C}{2} = \frac{IM}{IN} = \frac{r}{IN}$$

and, with the help of equation (2),

$$IN = \frac{r}{\sin \frac{C}{2}} = IC. \quad (3)$$

Because $\angle PIC$ is an exterior angle of $\Delta AIC$, $\angle PIC = \frac{A}{2} + \frac{C}{2} = 90^\circ - \frac{B}{2}$, whence

$$\cos \angle PIC = \sin \frac{B}{2}. \quad (4)$$

Putting the pieces together, we deduce

$$PN \parallel BM \iff \frac{IM}{IB} = \frac{IP}{IN} \iff \sin \frac{B}{2} = \frac{IM}{IB} = \frac{IP}{IC} \quad (\text{from } (1) \text{ and } (3)) \iff \frac{IP}{IC} = \cos \angle PIC \quad (\text{from } (4)) \iff PC \perp AI.$$
3920. Proposed by Alina Sîntămârian.

Evaluate

\[ \sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n + 2)!}. \]

There were 15 submitted solutions for this problem, 14 of which were correct. We present three solutions, representative of the two main solution methods utilized together with one variant.

Solution 1, by the AN-anduud Problem Solving Group.

Consider the following two power series,

\[ \sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad \text{and} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}. \]

Hence, we have

\[ \sin 1 = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} = \sum_{n=0}^{\infty} \left( \frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right), \]

and

\[ e = \sum_{n=1}^{\infty} \frac{1}{n!}. \]

Using the above considerations, we get

\[ \sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n + 2)!} = \sum_{n=0}^{\infty} \frac{(4n + 2)(4n + 1) + 2(4n + 2) + 1}{(4n + 2)!} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \left( \frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right) \]

\[ = e + \sin 1. \]

Solution 2, by the group of Dionne Bailey, Elsie Campbell, and Charles Diminnie.

To begin, we note that for \( n \geq 0 \),

\[ \frac{16n^2 + 20n + 7}{(4n + 2)!} = \frac{(4n + 2)(4n + 1) + 2(4n + 2) + 1}{(4n + 2)!} \]

\[ = \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!}. \]

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and hence,
\[ \sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n+2)!} = \sum_{n=0}^{\infty} \frac{1}{(4n)!} + 2 \sum_{n=0}^{\infty} \frac{1}{(4n+1)!} + \sum_{n=0}^{\infty} \frac{1}{(4n+2)!} \]
(since the Ratio Test easily confirms that each of the three series on the right converges).

The remainder of this solution depends on the following known series:
\[
\begin{align*}
\sin 1 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}, & \quad \cos 1 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}, \\
\sinh 1 &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}, & \quad \cosh 1 &= \sum_{k=0}^{\infty} \frac{1}{(2k)!},
\end{align*}
\]

Since we have
\[
(-1)^k + 1 = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad (-1)^{k+1} + 1 = \begin{cases} 0 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd} \end{cases},
\]
we obtain:
\[
\begin{align*}
\sin 1 + \sinh 1 &= \sum_{k=0}^{\infty} \frac{(-1)^k + 1}{(2k+1)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n)]!} = 2 \sum_{n=0}^{\infty} \frac{1}{(4n+1)!}, \\
\cos 1 + \cosh 1 &= \sum_{k=0}^{\infty} \frac{(-1)^k + 1}{(2k)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n)]!} = 2 \sum_{n=0}^{\infty} \frac{1}{(4n)!}, \\
-\cos 1 + \cosh 1 &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} + 1}{(2k)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n+1)]!} = 2 \sum_{n=0}^{\infty} \frac{1}{(4n+2)!}.
\end{align*}
\]

Therefore, we obtain,
\[
\sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n+2)!} = \frac{\cos 1 + \cosh 1}{2} + (\sin 1 + \sinh 1) + \frac{-\cos 1 + \cosh 1}{2}
\]
\[
= \sin 1 + \sinh 1 + \cosh 1 \\
= \sin 1 + \frac{e - e^{-1}}{2} + \frac{e + e^{-1}}{2} \\
= \sin 1 + e.
\]

*Solution 3, by Paolo Perfetti.*

First, we have:
\[
\frac{16n^2 + 20n + 7}{(4n+2)!} = \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!}.
\]
Let
\[ f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}, \]
so that we obtain:
\[ f'(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!}, \quad f''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!}, \quad f'''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)!}, \]
\[ f^{iv}(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} = f(x). \]
Thus \( f(x) \) satisfies \( f^{iv}(x) = f(x), \ f(0) = 1, \ f'(0) = 0, \ f''(0) = 0, \ f'''(0) = 0, \)
whose unique solution is \( f(x) = \frac{1}{2} \cosh x + \frac{1}{2} \cos x. \) Evaluating, we get
\[ f(1) = \frac{1}{2} \cosh 1 + \frac{1}{2} \cos 1 = \sum_{n=0}^{\infty} \frac{1}{(4n)!}. \]
Moreover, if we define
\[ g(x) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!}, \]
we get \( g(1) = \sum_{n=0}^{\infty} \frac{1}{(4n+1)!} \) and \( g'(x) = f(x), \ g(0) = 0. \) This implies
\[ g(x) = \frac{1}{2} \sinh x + \frac{1}{2} \sin x, \quad g(1) = \frac{1}{2} \sinh 1 + \frac{1}{2} \sin 1. \]
Finally, defining
\[ h(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}, \]
we get \( h(1) = \sum_{n=0}^{\infty} \frac{1}{(4n+2)!} \) and \( h'(x) = g(x), \ h(0) = 0. \) This implies
\[ h(x) = \frac{1}{2} \cosh x - \frac{1}{2} \cos x, \quad h(1) = \frac{1}{2} \cosh 1 - \frac{1}{2} \cos 1. \]
Summing up the terms, we obtain
\[ f(1) + 2g(1) + h(1) = e + \sin 1. \]

*Editor’s Comment.* The presented solutions illustrate three techniques: rearrange
the summations wisely to get a simple expression, rearrange the summations and
then recall other atypical power series that make things work, and solve a couple of
DEs to avoid having to work too much with power series. Wagon commented that
the sum can be explicitly computed when the numerator is an arbitrary quadratic
in \( n. \)