OLYMPIAD SOLUTIONS


**OC156.** Let $ABCD$ be a tetrahedron. Prove that vertex $D$, center of insphere and centroid of $ABCD$ are collinear if and only if the areas of triangles $ABD$, $BCD$, $CAD$ are equal.

*Originally question 2 from day 1 of the Poland Math Olympiad.*

We received four correct solutions to this problem. We present the solution by Michel Bataille.

Let $I$ and $r$ be the center and the radius of the insphere. Let $V(\cdot)$ and $A(\cdot)$ denote volume and area, respectively.

Since $V(IBCD) = \frac{1}{3} \cdot r \cdot A(BCD)$ (and similarly for triangles $CDA$, $DAB$ and $ABC$), we can use

$$(A(BCD) : A(CDA) : A(DAB) : A(ABC))$$

instead of

$$V(IBCD) : V(ICDA) : V(IDAB) : V(IABC))$$

for the barycentric coordinates of $I$ relative to $(A, B, C, D)$. It follows that

$$\sigma I = (A(BCD))A + (A(CDA))B + (A(DAB))C + (A(ABC))D$$

where $\sigma$ is the sum of the areas of the faces of $ABCD$ and the bold face letters represent the coordinates of the points. In particular, we have

$$\sigma \overrightarrow{DI} = (A(BCD))\overrightarrow{DA} + (A(CDA))\overrightarrow{DB} + (A(DAB))\overrightarrow{DC}. \quad (1)$$

Let $G$ be the centroid of $ABCD$. Then,

$$4G = A + B + C + D,$$

from which we deduce

$$4 \overrightarrow{DG} = \overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC}. \quad (2)$$

Now, $D, I, G$ are collinear if and only if

$$\sigma \overrightarrow{DI} = \lambda (4 \overrightarrow{DG}) \quad (3)$$

for some real number $\lambda$. Since $\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}$ are not coplanar, (1) and (2) show that (3) occurs if and only if

$$A(BCD) = A(CDA) = A(DAB).$$

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OC157. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
\[
f(f(x)^2 + f(y)) = xf(x) + y, \forall x, y \in \mathbb{R}.
\]

Originally question 4 from Kyrgyzstan National Olympiad.

We received three correct solutions and one incorrect submission. We present the solution by Henry Ricardo.

Letting $x = 0$ in the given equation, we have
\[
f(f(0)^2 + f(y)) = y, \forall y \in \mathbb{R}
\]
so $f$ is onto. Thus there exists $a \in \mathbb{R}$ such that $f(a) = 0$. Taking $x = a$ in the original equation gives us
\[
f(f(y)) = y, \forall y \in \mathbb{R},
\]
which shows that $f$ is one to one since $f(x) = f(y)$ implies $f(f(x)) = f(f(y))$, or by (2), $x = y$.

Now replace $x$ by $f(x)$ in the original equation to obtain
\[
f(x^2 + f(y)) = f(x)f(f(x)) + y = xf(x) + y = f(f(x)^2 + f(y))
\]
Usning the fact that $f$ is one to one, we have
\[
x^2 + f(y) = f(x)^2 + f(y).
\]
Therefore $x^2 = f(x)^2$ and $f(x) = \pm x$.

To eliminate the possibility that $f(x) = x$ for some $x$ and $f(y) = -y$ for some $y \neq x$, suppose that $xy \neq 0$ and $f(x) = x, f(y) = -y$. The original equation gives us $f(x^2 - y) = x^2 + y$, but we know that $f(x) = \pm x$ and so $\pm(x^2 - y) = x^2 + y$ implies either $x = 0$ or $y = 0$. Now it is clear that $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x$ for all $x \in \mathbb{R}$ satisfy the original equation and are the only such functions.

OC158. Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

Originally question 4 from day 1 of the Romania TST.

We received no solutions to this problem.

OC159. Let $p$ be an odd prime number. Prove that there exists a natural number $x$ such that $x$ and $4x$ are both primitive roots modulo $p$.

Originally question 3 from the 2012 Iran National Math Olympiad Third Round.

We received one correct solution and one incorrect submission. We present the solution by Oliver Geupel.
The existence of a primitive root modulo $p$ is a well-known fact. Suppose that $a$ is a primitive root modulo $p$. Then there is a positive integer $r$ such that $2 \equiv a^r \pmod{p}$ and therefore $4 \equiv a^{2r} \pmod{p}$. Let $p_1, p_2, \ldots, p_\ell$ denote the distinct prime divisors of $p - 1$. For $1 \leq k \leq \ell$, let $s_k$ be an integer such that $s_k \not\equiv 0 \pmod{p_k}$ and $s_k \not\equiv -2r \pmod{p_k}$. Find via the Chinese Remainder Theorem, a natural number $m$ such that

$$m \equiv s_k \pmod{p_k}, \quad 1 \leq k \leq \ell.$$ 

Then, neither $m$ nor $m + 2r$ is divisible by any $p_k$. Hence, each of the numbers $m$ and $m + 2r$ is coprime with $p - 1$.

We obtain

$$p - 1 \nmid m, \ 2m, \ 3m, \ldots, \ (p - 2)m,$$

$$p - 1 \nmid m + 2r, \ 2(m + 2r), \ 3(m + 2r), \ldots, \ (p - 2)(m + 2r).$$

Thus,

$$a^m, \ a^{2m}, \ a^{3m}, \ldots, \ a^{(p-2)m} \not\equiv 1 \pmod{p},$$

$$a^{m+2r}, \ a^{2(m+2r)}, \ a^{3(m+2r)}, \ldots, \ a^{(p-2)(m+2r)} \not\equiv 1 \pmod{p},$$

since $a$ is a primitive root modulo the prime $p$. We have obtained that $a^m$ and $a^{m+2r} \equiv 4a^m \pmod{p}$ are primitive roots modulo $p$. Therefore, $x = a^m$ has the required property.

**OC160.** The incircle of triangle $ABC$, is tangent to sides $BC, CA$ and $AB$ at $D, E$ respectively $F$. Let $T$ and $S$ be the reflection of $F$ with respect to $B$ respectively the reflection of $E$ with respect to $C$. Prove that the incenter of triangle $AST$ is inside or on the incircle of triangle $ABC$.

*Originally question 3 from day 2 of the Iran National Math Olympiad Second Round.*

*No solutions were received.*