THE CONTEST CORNER
No. 32
Robert Dawson

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n’importe quel problème. S’il vous plaît vous référer aux règles de soumission à l’endos de la couverture ou en ligne.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le 1 avril 2016 ; toutefois, les solutions reçues après cette date seront aussi examinées jusqu’au moment de la publication.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d’avoir traduit les problèmes.

CC156. Décrire et réaliser un croquis précis de la région qui représente l’ensemble
\{(x, y, z) : |x| + |y| ≤ 1, |y| + |z| ≤ 1, |z| + |x| ≤ 1\}.

CC157. Étant donné une matrice 5 × 5 dont chacun des nombres est un 0 ou un 1, démontrer qu’il doit exister une sous-matrice 2 × 2 (c’est-à-dire l’intersection de la réunion de deux rangées avec la réunion de deux colonnes) dont tous les nombres sont soit 0, soit 1.

CC158. On considère un point mobile A sur la partie positive de l’axe des abscisses, un point mobile B sur la partie positive de l’axe des ordonnées et l’origine O de manière que le triangle ABO ait toujours une aire de 4. Déterminer l’équation d’une courbe, définie dans le premier quadrant, qui est tangente à chacun des segments AB.

CC159. La disposition de huit tuiles carrées, ci-dessous à gauche, peut être divisée en deux groupes congruents de quatre tuiles, comme sur la droite. (On remarque qu’un groupe est le reflet de l’autre dans un miroir, ce qui est permis.)

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

Déterminer une façon de diviser la disposition suivante de 100 tuiles en deux

(48/ THE CONTEST CORNER)
groupes congruents de 50 tuiles, ou démontrer qu’il est impossible de le réaliser.

... ... ...
■ ■ ■ □ □ □
(19 tuiles dans la 10e rangée)

CC160. Déterminer tous les triplets \((f,g,h)\) de fonctions continues à valeurs réelles définies sur \(\mathbb{R}\) telles pour tout nombre réel \(x\),

\[ f(g(x)) = g(h(x)) = h(f(x)) = x. \]

CC156. Describe and accurately sketch the region

\[ \{(x, y, z) : |x| + |y| \leq 1, |y| + |z| \leq 1, |z| + |x| \leq 1\}. \]

CC157. Show that if a \(5 \times 5\) matrix is filled with zeros and ones, there must always be a \(2 \times 2\) submatrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

CC158. Suppose movable points \(A, B\) lie on the positive \(x\)-axis and \(y\)-axis, respectively, in such a way that \(\triangle ABO\), where \(O\) is the origin, always has area 4. Find an equation for a curve in the first quadrant which is tangent to each of the line segments \(AB\).

CC159. The following pattern of eight square tiles can be divided into two congruent sets of four tiles as shown. (Note that one set is the mirror image of the other — this is legal.)

Copyright © Canadian Mathematical Society, 2015
Find a way to divide the following pattern of 100 tiles into two congruent sets of fifty tiles, or show it cannot be done.

![Pattern of 100 tiles]

(19 tiles in the 10th row)

**CC160.** Find all triples of continuous functions \( f, g, h : \mathbb{R} \mapsto \mathbb{R} \) such that, for all \( x \in \mathbb{R} \),

\[
f(g(x)) = g(h(x)) = h(f(x)) = x.
\]

---

**Three really magic squares**

Having received his yearly salary in silver coins, the royal Mathematician arranged the coins into vertical stacks and placed them on a \( 3 \times 3 \) square so that the numbers representing the amount of coins in each stack formed a magic square, that is a square such that the sum of the numbers along every row, column and diagonal of the square is the same. Some stacks came out being quite tall, but none were higher than 300 coins tall.

The King liked the arrangement but lamented over the fact that all the numbers came out being composite. “If your majesty gives me 9 more coins, I will add one to each stack; the magic square property will be preserved but all the new numbers will be prime”, replied the Mathematician. The King nearly agreed, but was interrupted by the Joker, who took away one coin from each stack and the new numbers all became prime (and the square, of course, remained a magic square).

What was the original magic square composed by the Mathematician?

*From Kvant, 1981 (9), p.31.*
CONTEST CORNER
SOLUTIONS


CC106. At each summit of a regular tetrahedron of side length 3, we cut off a pyramid such that the cut-off surface makes an equilateral triangle. The four equilateral triangles thus obtained have all different dimensions. What is the total length of the edges of the solid thus truncated? Provide a proof.

Originally problem 29 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2008.

We received two incomplete submissions to this problem, neither of which adequately proved that the four tetrahedral corners that are removed from the original tetrahedron must be regular. We present an editor’s solution.

Let $A, B, C$, and $D$ be the vertices of the given tetrahedron, and let $P, Q$, and $R$ be the points on $DA, DB$, and $DC$, respectively, such that $PQR$ is the equilateral triangle formed by cutting off the pyramid containing the vertex $D$. Suppose that the polyhedron has been labeled so that $DP \geq DQ \geq DR$. We will first prove that these segments must, in fact, be equal. Compare triangles $DPR$ and $DQR$. We have $RP = RQ$ and the angles at $D$ are both $60^\circ$. Because we assume that $RD$ is no larger than either $DP$ or $DQ$, the angles at $P$ and $Q$ must be acute.

From the sine law (applied to both triangles) we have

$$\sin \angle DQR = DR \frac{\sin 60^\circ}{RQ} = DR \frac{\sin 60^\circ}{RP} = \sin \angle DPR,$$

from which we conclude that the two triangles are congruent, whence $DQ = DP$.

Focusing now on the $60^\circ$ angle $PDQ$, we note that the length of the segment $PQ$ increases monotonically as the lengths $DP = DQ$ increase, so there will be exactly one position of $P$ and $Q$ for which $PQ = QR$ ($= RP$), namely where the lengths $DP, DQ, DR$ are all equal. Because the angles at $D$ are all $60^\circ$, the three faces at $D$ are equilateral triangles, and the tetrahedron $DPQR$ is therefore regular.

Returning to the problem, because all edges of a regular tetrahedron have the same length, $DP + DQ + DR = PQ + QR + RP$ and we conclude that the truncation at the vertex $D$ does not change the sum of the edge lengths. Of course, the same can be said about the truncation at the other vertices, so the total length of the edges of the truncated tetrahedron must equal 18.

CC107. In a right triangle $ABC$ with right angle at $B$ and $BC = 1$, we place $D$ on side $AC$ such that $AD = AB = \frac{1}{2}$. What is the length of $DC$?

Copyright © Canadian Mathematical Society, 2015
Originally problem 6 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2002.

There were eight solution submitted for this question, all essentially the same.

By the Pythagorean Theorem we have
\[ AC = \sqrt{BC^2 + AB^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2} \]
and
\[ DC = AC - AD = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5} - 1}{2}. \]

CC108. In an orthonormal system, the line with equation \( y = 5x \) crosses the parabola with equation \( y = x^2 \) in point \( A \). The perpendicular to \( OA \) at \( O \) intersects the parabola at \( B \). What is the area of triangle \( AOB \)?

Originally problem 20 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.

We received six correct solutions, and one incorrect solution. We present the solution of Titu Zvonaru.

It is easy to deduce that \( A(5,25) \). The slope of \( OB \) is \(-\frac{1}{5}\). Solving the system \( y = -\frac{1}{5}x, y = x^2 \) we obtain \( B(-\frac{1}{5}, \frac{1}{25}) \).

It follows that \( OA = \sqrt{5^2 + 25^2} = 5\sqrt{26}, OB = \sqrt{\frac{1}{5^2} + \frac{1}{25^2}} = \frac{\sqrt{26}}{25}. \) Hence the area of the triangle is \( AOB = \frac{OA \cdot OB}{2} = \frac{26}{10} = \frac{13}{5}. \)

CC109. Let \( E \) be the set of reals \( x \) for which the two sides of the following equality are defined:
\[ \cot 8x - \cot 27x = \frac{\sin kx}{\sin 8x \sin 27x}. \]

If this equality holds for all the elements of \( E \), what is the value of \( k \)?

Originally problem 21 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.

We received seven submitted solutions to this problem, one of which was incorrect and five were incomplete. We present the only correct solution by Paolo Perfetti modified by the editor.

Note first that \( E = \{ x \in \mathbb{R} | x \neq \frac{m\pi}{8} \text{ and } x \neq \frac{m\pi}{27} \} \) for any \( m \in \mathbb{Z} \). For \( x \in E \), the given equality is equivalent to
\[ \sin 8x \cdot \sin 27x(\cot 8x - \cot 27x) = \sin kx. \]  \hspace{1cm} (1)

We shall prove that the only value of \( k \) for which (1) holds for all \( x \in E \) is \( k = 19 \).
Since
\[
\sin 8x \cdot \sin 27x (\cot 8x - \cot 27x) = \sin 27x \cos 8x - \cos 27x \sin 8x = \sin (27x - 8x) = \sin 19x,
\]
k = 19 satisfies (1).

Next, suppose (1) holds for all \(x \in E\) and some \(k \in \mathbb{Z}\) with \(k \neq 19\).
If \(k = -19\), then from (1) we have \(2 \sin 19x = 0\) for all \(x \in E\), which is false (for example, if \(x = \frac{\pi}{38}\), then \(x \in E\), but \(\sin 19x = \sin \frac{\pi}{2} = 1 \neq 0\)). Hence \(k \neq -19\).

From (1), we also have
\[
2 \sin \left(\frac{19 - k}{2}x\right) \cos \left(\frac{19 + k}{2}x\right) = 0. \tag{2}
\]
Since \(\sin \left(\frac{19-k}{2}x\right) = 0\) if and only if \(\frac{19-k}{2}x = m\pi\) or \(x = \frac{2m\pi}{19-k}\) and \(\cos \left(\frac{19+k}{2}x\right) = 0\) if and only if \(\frac{19+k}{2}x = (m + \frac{1}{2})\pi\) or \(x = \frac{(2m+1)\pi}{19+k}\) for some \(m \in \mathbb{Z}\), there must be some \(x \in E\) that does not satisfy (2). (To be more precise, the set of all \(x\) such that \(x = \frac{2m\pi}{19-k}\) or \(x = \frac{(2m+1)\pi}{19+k}\) for some \(m \in \mathbb{Z}\) is countable while \(E\) is clearly uncountable.) This is a contradiction and our proof is complete.

**CC110.** What is the number of real solutions to the equation:
\[
|1 + x - |x - 1| - |1 - x|| = |-x - |x - 1||.
\]

*Originally problem 26 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.*

We have received four correct solutions and one incorrect submission. We present the solution by Henry Ricardo.

We compute the left-hand side (LHS) and the right-hand side (RHS) on three intervals that cover the real number line.

*Case 1.* Suppose that \(0 \leq x \leq 1\). Then
\[
\text{RHS} = |-x - (1-x)| = |x - 1 + x| = 1.
\]

When \(x \in [-\frac{1}{2}, 0]\),
\[
|1 + x - |x - (1-x)|| = |1 + x - (1 - 2x)| = 3x
\]
and when \(x \in (\frac{1}{2}, 1]\),
\[
|1 + x - |2x - 1|| = |1 + x - (2x - 1)| = |2 - x| = 2 - x
\]
so that
\[
\text{LHS} = \begin{cases} 
3x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2 - x & \text{if } \frac{1}{2} < x \leq 1.
\end{cases}
\]

Copyright © Canadian Mathematical Society, 2015
Thus LHS = RHS when either $3x = 1$ or $2 - x = 1$, which implies $x = \frac{1}{3}$ and $x = 1$ for $x \in [0, 1]$.

**Case 2.** If $x > 1$, we have

$$LHS = |1 + x - |x - (x - 1)|| = |1 + x - 1| = x$$

and

$$RHS = |-x - (x - 1)| = |-2x + 1| = 2x - 1.$$  

But $x > 1$ implies that $(2x - 1) - x = x - 1 > 0$, so $RHS > LHS$ and there are no solutions to the equation in the interval $(1, \infty)$.

**Case 3.** Finally, for $x \in (-\infty, 0)$, $RHS = 1$ and $LHS = |3x| = -3x$, so $LHS = RHS$ if and only if $-3x = 1$, which implies $x = -\frac{1}{3}$.

Thus the only solutions of the given equation are $x = -\frac{1}{3}, 1, \frac{1}{3}$.

---

**Math Quotes**

The solution of problems is one of the lowest forms of mathematical research. Yet, its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

*Benjamin Franklin Finkel.*

---

*Crux Mathematicorum, Vol. 41(2), February 2015*