A Mathematical Performance (II)

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*Part I of this article was published in **Crux** 41(9), p. 392–396. The three authors participated in 2010* International Mathematics Competition (IMC)*. Below, we present the contest rules as well as the 2010 individual and team contest papers. For further details, see their website* http://www.imc-official.org/en_US/

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The twelve questions in the individual contest and the five questions in the team contest require answers only. The three problems in the individual contest and the five problems in the team contest require full solutions.

Two hours are allowed for the individual contest, and one hour for the team contest. In the first 10 minutes of the team contest, the four team members examine Part I together, with no writing allowed. During this time, they divide up the questions and problems among themselves in any way they wish, and then go their separate ways. In the next 35 minutes, they write down the answers to the questions and the solutions to the problems in their shares. In the final 15 minutes, the team members reconvene and work on Part II together.

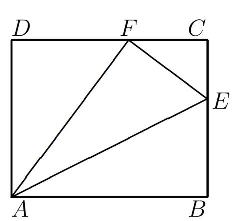
Individual Contest

Question 1.
Let p, q and r be real numbers such that $p+q+r = 26$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 31$. What is the value of $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}+\frac{q}{p}+\frac{r}{q}+\frac{p}{r}$?

Question 2.
At a charity dinner, each person consumed half a plate of rice, a third of a plate of vegetables and a quarter of a plate of meat. Overall, 65 plates of food were served. What was the number of people at the charity dinner ?

Question 3.
What is the number of triples (x, y, z) of positive integers which satisfy $xyz = 3^{2010}$ and $x \le y \le z < x + y$?

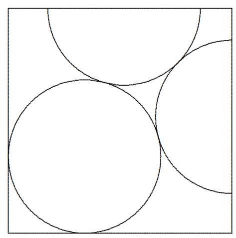
Question 4.
E is a point on the side BC of a rectangle $ABCD$ such that if a fold is made along AE, as shown in the diagram below, the vertex B coincides with a point F on the side CD. If $AD = 16$ cm and $BE = 10$ cm, what is the length, in cm, of AE ?



Question 5.
What is the smallest four-digit number which has exactly 14 positive divisors (including 1 and itself), such that the units digit of one of its prime divisors is 3 ?

Question 6.
Let $f(x)$ be a fourth degree polynomial. What is the remainder when $f(2010)$ is divided by 10 if $f(1) = f(2) = f(3) = 0$, $f(4) = 6$ and $f(5) = 72$?

Question 7.
A circle and two semicircles, all of radius 1 cm, touch one another inside a square, as shown in the diagram below. What is the area, in cm^2 of the square ?



Question 8.
Let $p \geq q$ be prime numbers such that $p^3 + q^3 + 1 = p^2q^2$. What is the maximum value of $p + q$?

Question 9.
The sum of n positive integers, not necessarily distinct, is 100. The sum of any 7 of them is less than 15. What is the minimum value of n ?

Question 10.
P is a point inside triangle ABC such that $\angle ABP = 20^\circ$, $\angle PBC = 10^\circ$, $\angle ACP = 20^\circ$ and $\angle PCB = 30^\circ$. What is the measure, in degrees, of $\angle CAP$?

Question 11.
A farmer puts 100 chickens and 100 pigs into four enclosures in two rows and two columns. There are 120 heads in the first row, 300 legs in the second row, 100 heads in the first column and 320 legs in the second column. How many different ways can this be done ?

Question 12.
An animal shelter consists of five cages in a row, labelled from left to right as shown in the diagram below. There is one animal in each cage.

Red Wolf	Silver Lion	Brown Fox	White Cow	Gray Horse

The five animals are indeed a wolf, a lion, a fox, a cow and a horse, and their colours are indeed red, silver, brown, white and gray. However, none of the labels matches any of the animals. Moreover, no animal is in or next to a cage whose label either matches its type or its colour. If the horse is not in the middle cage, what is the colour of the horse ?

Problem 13.
A segment divides a square into two polygons each of which has an incircle. One of the circles has radius 6 cm while the other one is larger. What is the difference, in cm, between twice the length of this segment and the side length of the square ?

Problem 14.
A small bag of candy contains 6 pieces. A medium bag of candy contains 9 pieces. A large bag of candy contains 20 pieces. If we buy candy in bags only, what is the largest number of pieces which we cannot obtain exactly ?

Problem 15.
For any positive integer n, S_n is the sum of the first n terms of a given sequence $a_1, a_2, a_3, \ldots$ where $a_1 = 2010$. If $S_n = n^2 a_n$ for every n, what is the value of a_{2010} ?

Team Contest

Part I.

Question 1.
Solve the following system of equations for real numbers $w,\ x,\ y$ and z :

$$\begin{aligned} w + 8x + 3y + 5z &= 20; \\ 4w + 7x + 2y + 3z &= -20; \\ 6w + 3x + 8y + 7z &= 20; \\ 7w + 2x + 7y + 3z &= -20. \end{aligned}$$

Problem 2.
In the convex quadrilateral $ABCD$, AB is the shortest side and CD is the longest. Prove that $\angle A > \angle C$ and $\angle B > \angle D$.

Question 3.
Let $m \geq n$ be integers such that $m^3 + n^3 + 1 = 4mn$. Determine the maximum value of $m - n$.

Problem 4.
Arranged in an 8×8 array are 64 dots. The distance between adjacent dots on the same row or column is 1 cm. Determine the number of rectangles of area 12 cm^2 having all four vertices among these 64 dots.

Question 5.
Determine the largest positive integer n such that there exists a unique positive integer k satisfying $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$.

Problem 6.
In each row and each column of a 9×9 table with 81 numbers, at most four different numbers appear. What is the maximum number of different numbers that can appear in this table ?

Question 7.
In the convex quadrilateral $ABCD$, we have $\angle ADB = 16°$, $\angle BDC = 48°$, $\angle ACD = 58°$ and $\angle BCA = 30°$. Determine the measure, in degrees, of $\angle ABD$.

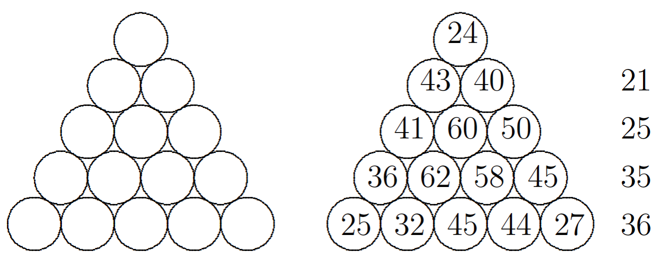
Problem 8.
Determine all ordered triples (x, y, z) of positive real numbers such that each of $x + \frac{1}{y}$, $y + \frac{1}{z}$ and $z + \frac{1}{x}$ is an integer.

Part II.

Question 9.
Put each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 into a different one of the fifteen circles in the diagram below on the left, so that

(1) for each circle, the sum of the numbers in it and in all circles touching it is as given by the diagram below on the right ;

(2) for each row except the first, the sum of the numbers in the circles on it is as given by the diagram below on the right.



Problem 10.
The letters K, O, R, E, A, I, M and C are written in eight rows, with 1 K in the first row, 2 Os in the second row, and so on, up to 8 Cs in the last row. Starting with the lone K at the top, try to spell the words KOREAIMC by moving from row to row, going to the letter directly below or either of its neighbours, as illustrated by the path in boldface. It turns out that one of these 36 letters may not be used. As a result, the total number of ways of spelling KOREAIMC drops to 516. Determine the letter which may not be used.

K							
O	O						
R	**R**	R					
E	**E**	E	E				
A	A	**A**	A	A			
I	I	I	**I**	I	I		
M	M	**M**	M	M	M	M	
C	C	C	**C**	C	C	C	C