SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(1), p. 28–31.

3901. *Proposed by D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Let $A, B \in M_n(\mathbb{R})$ with $\det A = \det B \neq 0$. If $a, b \in \mathbb{R} \setminus \{0\}$, prove that

$$\det\left(aA + bB^{-1}\right) = \det\left(aB + bA^{-1}\right).$$

We received 15 correct solutions, as well as one incorrect and one incomplete solution. We present two solutions.

Solution 1, by Dhananjay Mehendale; most of the received solutions were variations on this theme.

Since $\det A = \det B \neq 0$, we have

$$\det(A^{-1}) = (\det A)^{-1} = (\det B)^{-1} = \det(B^{-1}).$$

We use the fact that if $X, Y \in M_n(\mathbb{R})$, then $\det(XY) = \det X \cdot \det Y$. Let I be the identity matrix. Then the identity $\det(aAB + bI) = \det(aAB + bI)$ gives

$$\begin{aligned}\det(A^{-1}) \cdot \det(aAB + bI) &= \det(aAB + bI) \cdot \det(B^{-1}) \Leftrightarrow \\ \det\left(A^{-1}(aAB + bI)\right) &= \det\left((aAB + bI)B^{-1}\right) \Leftrightarrow \\ \det(aB + bA^{-1}) &= \det(aA + bB^{-1}),\end{aligned}$$

as was to be proved.

Solution 2, by Michel Bataille, slightly modified by the editor.

If I_n denotes the unit matrix of size n, we have

$$aA + bB^{-1} = aA\left(I_n + \frac{b}{a}A^{-1}B^{-1}\right) \quad \text{and} \quad aB + bA^{-1} = aB\left(I_n + \frac{b}{a}B^{-1}A^{-1}\right).$$

It follows that

$$\det(aA + bB^{-1}) = a^n \det(A) \cdot \det\left(I_n + \frac{1}{a}A^{-1} \cdot bB^{-1}\right)$$

and

$$\det(aB + bA^{-1}) = a^n \det(B) \cdot \det\left(I_n + bB^{-1} \cdot \frac{1}{a}A^{-1}\right).$$

The desired result now follows from $\det A = \det B$ and the general property : if $C, D \in M_n(\mathbb{R})$, then $\det(I_n + CD) = \det(I_n + DC)$. To prove this latter equality, let

$$E = \left(\begin{array}{c|c} I_n & -C \\ \hline D & I_n \end{array}\right) \qquad \text{and} \qquad F = \left(\begin{array}{c|c} I_n & O_n \\ \hline -D & I_n \end{array}\right),$$

where E, F are block-partitioned and O_n is the zero matrix of size n. Then,

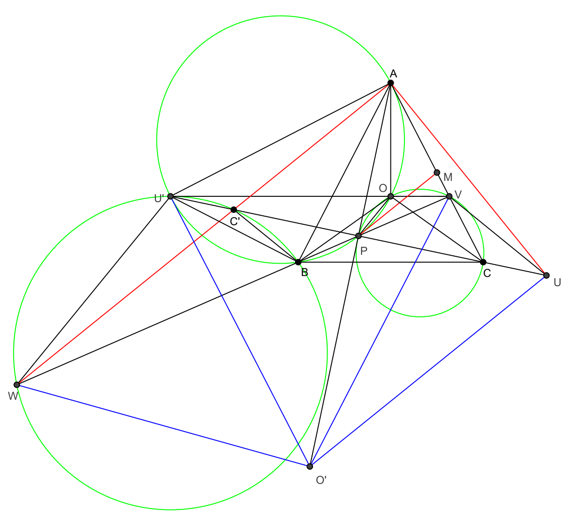
$$EF = \left(\begin{array}{c|c} I_n + CD & -C \\ \hline O_n & I_n \end{array}\right) \qquad \text{and} \qquad FE = \left(\begin{array}{c|c} I_n & -C \\ \hline O_n & I_n + DC \end{array}\right).$$

From properties of determinants, $\det(EF) = \det(FE)$. On the other hand, from the block matrix decomposition above we have $\det(EF) = \det(I_n + CD)$ and $\det(FE) = \det(I_n + DC)$, whence $\det(I_n + CD) = \det(I_n + DC)$ as claimed.

3902. *Proposed by Michel Bataille.*

Let ABC be a triangle with $AB = AC$ and $\angle BAC \neq 90°$ and let O be its circumcentre. Let M be the midpoint of AC and let P on the circumcircle of $\triangle AOB$ be such that $MP = MA$ and $P \neq A$. The lines l and m pass through A and are perpendicular and parallel to PM, respectively. Suppose that the lines l and PC intersect at U and that the line PB intersect AC at V and m at W. Prove that U, V and W are not collinear and that l is tangent to the circumcircle of $\triangle UVW$.

We received four correct submissions. We present a somewhat expanded version of the solution by Glenier L. Bello-Burguet.



Let us first see why U, V, W must be distinct. Note that $P \neq A$ (given) and $P \neq C$ (because $P = C$ would place O on the circumcircle of ΔABC). Thus APC is a proper triangle, and because M is its circumcenter, it follows that $\angle CPA = 90°$. So W (on a line through A that is parallel to MP) cannot coincide with U (on the perpendicular through A to MP) or V (on AC). On the other hand, should $U = V$ then the lines BP and CP would coincide, which would imply that $P \in BC$; furthermore, the lines AC and AU would coincide and we would have $PM \perp AC$. As a consequence, PMA, PMC, and PCA would all be isosceles right triangles, in which case P would be the midpoint of BC, whence $\angle BAC = 90^\circ$. This case has been excluded from the problem.

Our next goal is to prove that $OV||BC$. We shall use directed angles, where $\angle XYZ$ denotes the angle through which the line YX must be rotated in the positive direction about Y to coincide with YZ. (Otherwise some of the angles involved in the proof would have to be replaced by their supplements, depending on whether the angle at A is obtuse or acute.) Using, in turn, that $OA = OC$, AO bisects $\angle BAC$, and the quadrilateral $BPOA$ is cyclic, we obtain

$$\angle VCO = \angle OAV = \angle BAO = \angle BPO = \angle VPO.$$

Hence $OPCV$ is cyclic. By using $MA = MP$, $BPOA$ is cyclic, $OA = OB$, AO bisects $\angle BAC$ and, again, $BPOA$ is cyclic, we also have

$$\begin{aligned}\angle MPO &= \angle MPA - \angle OPA = \angle PAM - \angle OPA = \angle PAM - \angle OBA \\ &= \angle PAM - \angle BAO = \angle PAM - \angle OAM = \angle PAO = \angle PBO.\end{aligned}$$

Thus (because $\angle VPO$ is an exterior angle of ΔBPO)

$$\angle BOP = \angle VPO - \angle PBO = \angle VPO - \angle MPO = \angle VPM, \qquad (1)$$

whence (using the circle $OPCV$ and $MP = MC$)

$$\begin{aligned}\angle BOV &= \angle BOP + \angle POV = \angle VPM + \angle POV = \angle VPM + \angle PCV \\ &= \angle VPM + \angle MPC = \angle VPC = \angle BPC.\end{aligned}$$

Hence,

$$\begin{aligned}\angle VOA &= \angle VOB + \angle BOA = \angle VOB + \angle BPA \\ &= \angle CPB + \angle BPA = \angle CPA.\end{aligned}$$

Because $\angle CPA = 90^\circ$, also $\angle VOA = 90^\circ$; that is, $AO \perp OV$ and (because in the isosceles triangle ABC we have $AO \perp BC$), we conclude that $OV \parallel BC$, as desired.

Let U' be the symmetric point of U with respect to P and let C' be the intersection of the lines UU' and $m = AW$. We will show that U' is the second intersection point of VO with the circumcircle of ΔAOB. Since $AC' \parallel PM$ and M is the midpoint of AC, we must have $C'P = PC$ and, therefore, $\angle U'AC' = \angle CAU$. Hence (because $\ell = AU$ is perpendicular to $m = AW$),

$$\angle U'AC = \angle C'AU = 90^\circ.$$

Since $\triangle ABC$ is isosceles and O is its circumcenter, we know that AC is tangent at A to the circumcircle of $\triangle ABO$ (because that angle between the chord AO and the line AC equals $\angle BAO$, which equals the inscribed $\angle OBA$ that is subtended by AO). Therefore, the center of circle passing through the points B, O, P and A lies on the line AU'. Note that $\angle APU' = \angle UPA = 90°$, so AU' must be a diameter of that circle. It follows that $\angle AOU' = 90°$ and, furthermore, since $AO \perp OV$, we conclude that U', O and V are collinear. Thus we see that U' lies, as claimed, on both the line VO and the circle $BPOA$. Moreover, we now have

$$U'V \parallel BC. \tag{2}$$

Let O' be the point where AP intersects the line perpendicular to AU at U. Our ultimate goal is to show that the circle of interest, namely UVW, has center O' and radius $O'U$. By symmetry with respect to P, we have $\angle O'U'A = \angle AUO' = 90°$, so (recalling that $U'AC = 90°$)

$$O'U' \parallel CA. \tag{3}$$

Using (2) and (3) we get

$$\frac{PB}{PV} = \frac{PC}{PU'} = \frac{PA}{PO'},$$

and therefore

$$VO' \parallel AB. \tag{4}$$

Now by (2), (3) and (4) we obtain that $\triangle U'O'V \sim \triangle CAB$. Since $AB = AC$ we get that $O'V = O'U'$. Furthermore, by symmetry about $O'P$ we have $O'U' = O'U$; in other words, O' is the center of the circle UVU'. It remains to show that W lies on this circle.

Using that $AW \parallel PM$ and (1) we have

$$\angle BWC' = \angle VPM = \angle BOP = \angle BU'P = \angle BU'C',$$

and, therefore, $WBC'U'$ is cyclic. On the other hand, using (2) together with the midpoint property of P, we get

$$\frac{PC'}{PU} = \frac{PC}{PU'} = \frac{PB}{PV},$$

so $UV \parallel BC'$. Hence

$$\angle U'UV = \angle PC'B = \angle U'C'B = \angle U'WB = \angle U'WV.$$

This implies that $UVU'W$ is cyclic, as claimed. Recalling that we defined $AU \perp O'U$, we conclude, finally, that AU is tangent to the circumcircle of ΔUVW. As a further consequence, we have proved that U, V and W are not collinear (since they are distinct points on a circle), and our proof is complete.

3903. *Proposed by George Apostolopoulos.*

Consider a triangle ABC with an inscribed circle with centre I and radius r. Let C_A, C_B and C_C be circles internal to ABC, tangent to its sides and tangent to the inscribed circle with the corresponding radii r_A, r_B and r_C. Show that $r_A + r_B + r_C \geq r$.

We received 18 correct submissions. We present the solution by Michel Bataille, which is similar to many other solutions received.

First, we remark that C_A is the image of C under the homothety with centre A and factor $\frac{r_A}{r}$, hence the centre I_A of C_A satisfies $\overrightarrow{AI_A} = \frac{r_A}{r}\overrightarrow{AI}$. Since we also have $II_A = r + r_A$, it follows that $\left(1 - \frac{r_A}{r}\right) AI = r + r_A$ and so

$$r_A = r \cdot \frac{AI - r}{AI + r} = r \cdot \frac{1 - \frac{r}{AI}}{1 + \frac{r}{AI}} = r \cdot \frac{1 - \sin\frac{A}{2}}{1 + \sin\frac{A}{2}}.$$

Similar results hold for r_B and r_C.
Now, let $f(x) = \frac{1-\sin x}{1+\sin x}$ $(x \in (0, \frac{\pi}{2}))$. An easy calculation gives $f''(x) = 2(1 + \sin x)^{-3}(\sin x + \cos^2 x + 1)$. Thus, $f''(x) > 0$ for all $x \in (0, \frac{\pi}{2})$ and f is convex on $(0, \frac{\pi}{2})$. From Jensen's inequality, we obtain

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq 3f\left(\frac{A+B+C}{6}\right) = 3 \cdot \frac{1 - \sin\frac{\pi}{6}}{1 + \sin\frac{\pi}{6}} = 1$$

and we can conclude

$$r_A + r_B + r_C = r\left(f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right)\right) \geq r.$$

Editor's comments. Bataille points out that this problem appeared in the Third Round of the Iranian Mathematical Olympiad 2002 and was previously solved in ***Crux*** [2006 : 373-374 ; 2007 : 350].

3904. *Proposed by Abdilkadir Altintaş.*

Let ABC be an equilateral triangle and let D, E and F be the points on the sides AB, BC and AC, respectively, such that $AD = 2$, $AF = 1$ and $FC = 3$. If the triangle DEF has minimum possible perimeter, what is the length of AE?

There were 20 correct solutions, with two from one solver. Thirteen exploited the reflection principle. Some used similar triangles to identify the position of E for minimum perimeter while others used vectors or analytic geometry. Five used the law of cosines to determine the side lengths of the triangle and minimized a function by differentiating. The featured solution follows the approach of the majority.

Since DF is fixed, the perimeter of the triangle is minimized when $DE + EF$ is minimized. Let G be the reflected image of F in the axis BC. Since $DE + EF = DE + EG$, by the reflection principle, the perimeter is minimized when D, E, G are collinear. In this situation, let x be the length of BE. Since $\angle DBE = \angle FCE = \angle ECG = 60^\circ$ and $\angle DEB = \angle GEC$, triangles BDE and CGE are similar. Therefore

$$\frac{x}{2} = \frac{BE}{BD} = \frac{CE}{CG} = \frac{4-x}{3},$$

whence $x = 8/5$. By the Law of Cosines applied to triangle ABE,

$$AE^2 = 16 + x^2 - 8x\cos 60^\circ = \frac{304}{25},$$

from which $AE = 4\sqrt{19}/5$.

The minimum perimeter turns out to be $\sqrt{3} + \sqrt{21}$.

3905. *Proposed by Jonathan Love.*

A sequence $\{a_n : n \geq 2\}$ is called *prime-picking* if, for each n, a_n is a prime divisor of n. A sequence $\{a_n : n \geq 2\}$ is called *spread-out* if, for each positive integer k, there is an index N such that, for $n \geq N$, the k consecutive entries $a_n, a_{n+1}, \cdots, a_{n+k-1}$ are all distinct. For example, the sequence

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

is spread-out. Does there exist a prime-picking spread-out sequence?

There were 2 submitted solutions for this problem, both incorrect. We present the proposer's solution, with minor clarifications.

The answer is yes. For each positive integer k and index n with $k! < n < (k+1)!$, define

$$b_n = \frac{n}{\gcd(k!, n)}.$$

Now, for any $n > k!$, suppose that p is a prime divisor of b_n. There is a non-negative exponent a for which $p^a < k \leq p^{a+1}$, so that p^{a+1} must divide n.

Then, suppose that $n > k!$, $i > 0$, and that p divides both b_n and b_{n+i}. We see that p^{a+1}, dividing both n and $n+i$, must divide i, so that $i \geq p^{a+1} > k$. It follows that the numbers $b_n, b_{n+1}, \ldots, b_{n+k-1}$ are pairwise coprime. Therefore, if we let a_n be any prime divisor of b_n, we obtain a prime-picking spread-out sequence, by the above arguments.

Editor's comments. The solutions to this problem illustrate a classic conundrum. One can pick the values a_n simply, and get one property essentially for free, and then work much harder to get the other property. Alternatively, one might work harder or more cleverly to choose the values a_n, and then work much less to obtain both properties.

The proposer's solution made a more complicated choice of a_n, but the work to obtain the requisite properties was minimal. The two submitted solutions made simpler choices of a_n, but more work was required to prove that the sequences were spread-out (since they were chosen to be trivially prime-picking), and both incorrect solutions contained errors in this work. That said, both choices of a_n were correct : letting a_n be either the largest prime divisor of n, or the prime divisor of n for which the corresponding prime power factor of n is largest, yields a prime-picking spread-out sequence.

3906★. *Proposed by Titu Zvonaru and Neculai Stanciu.*

If $x_1, x_2, \ldots, x_n$ are positive real numbers, then prove or disprove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

for all positive integers n.

We received one correct solution and one incorrect submission. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, modified slightly by the editor.

The proposed inequality is false in general.

Let $f(n)$ and $g(n)$ denote the expression on the left-hand side and the right-hand side of the given inequality, respectively. Then for $n = 15$ and $x_k = 2^{-k}$, $k = 1, 2, \ldots, n$, computations with the aid of a computer yield

$$f(15) = \left(\sum_{k=1}^{14} \frac{1}{2^{k-1}}\right) + \frac{1}{2^{29}} \approx 1.999938967,$$

$$g(15) = \sqrt{15 \sum_{k=1}^{15} \frac{1}{2^{2k}}} \approx 2.236067976,$$

showing that $f(15) < g(15)$.

Editor's comments. In general for the choice of x_k given above, note that

$$f(n) = \left(\sum_{k=1}^{n-1} \frac{1}{2^{k-1}}\right) + \frac{1}{2^{2n-1}} = \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} + \frac{1}{2^{2n-1}} = 2 - \frac{1}{2^{n-2}} + \frac{1}{2^{2n-1}} < 2$$

since $n - 2 < 2n - 1$. On the other hand, since

$$g(n) = \sqrt{n \sum_{k=1}^{n} \frac{1}{2^{2k}}} = \sqrt{\frac{n}{4} \sum_{k=1}^{n} \frac{1}{2^{2k-2}}} = \sqrt{\frac{n}{4} \cdot \frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)} = \sqrt{\frac{n}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)},$$

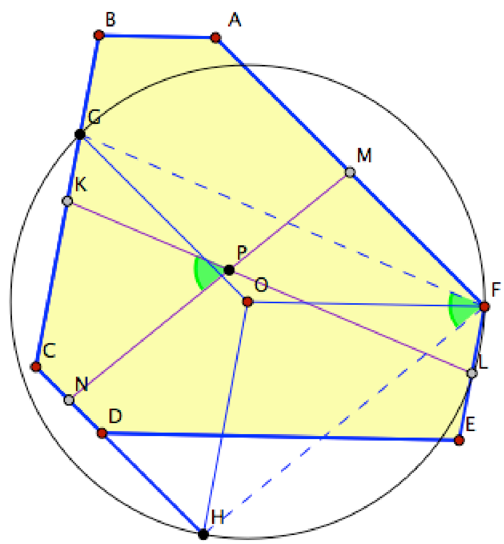
it is clear that $g(n)$ is an increasing function of n. Since we have already shown that $g(15) > 2$, it follows that $f(n) < g(n)$ for all $n \geq 15$. In fact, checking by computer reveals that the smallest n for which $f(n) < g(n)$ is $n = 12$.

3907. *Proposed by Enes Kocabey.*

Let $ABCDEF$ be a convex hexagon such that $AB + DE = BC + EF = FA + CD$ and $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Let the midpoints of the sides AF, CD, BC and EF be M, N, K and L, respectively, and let $MN \cap KL = \{P\}$. Show that $\angle BCD = 2\angle KPN$.

We received four submissions, three of which were correct and one incomplete. We present the solution by Oliver Geupel.

Remark. The problem is well known. It appeared on a test for the selection of the Taiwanese team for the IMO 2014. The problem can be found with solution at `http://www.artofproblemsolving.com/community/c6h598542p3551871.`
The following solution is essentially the same as the internet solution.



Let $2c$ be the common sum of the lengths of opposite sides. We define G and H to be the points where the halflines CB and CD meet the circle with centre C and radius c. In terms of the position vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$ corresponding to the points $A, B, C, \ldots$, we have

$$\vec{G} = \frac{1}{2}\left(\vec{B} + \vec{C} - \vec{E} + \vec{F}\right), \quad \text{and} \quad \vec{H} = \frac{1}{2}\left(-\vec{A} + \vec{C} + \vec{D} + \vec{F}\right).$$

Let O be the fourth vertex of the rhombus $CHOG$. That is,

$$\vec{O} = \frac{1}{2}\left(-\vec{A} + \vec{B} + \vec{D} - \vec{E} + 2\vec{F}\right).$$

These three assertions are easily verified :

$$2\overrightarrow{GO} = 2\left(\vec{O} - \vec{G}\right) = \vec{F} - \vec{A} + \vec{D} - \vec{C} = \overrightarrow{AF} + \overrightarrow{CD} = 2\left(\vec{H} - \vec{C}\right) = 2\overrightarrow{CH},$$

and, similarly,

$$2\overrightarrow{HO} = 2\overrightarrow{CG} = \overrightarrow{CB} + \overrightarrow{EF}.$$

Moreover, we also have

$$2\overrightarrow{FO} = 2\left(\vec{O} - \vec{F}\right) = \vec{B} - \vec{A} + \vec{D} - \vec{E} = \overrightarrow{AB} + \overrightarrow{ED}.$$

As a consequence, F lies along with G and H on the circle with centre O and radius c.

Furthermore,

$$\overrightarrow{FH} = \frac{1}{2}\left(-\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{F}\right) = \overrightarrow{MN}, \quad \text{and} \quad \overrightarrow{FG} = \overrightarrow{LK};$$

thus (because P lies on both MN and KL)

$$FH \parallel PN, \qquad \text{and} \qquad FG \parallel PK.$$

Putting everything together, we conclude

$$\angle BCD = \angle GCH = \angle GOH = 2\angle GFH = 2\angle KPN.$$

This completes the proof.

Editor's comments. A simple way to draw the required hexagon comes as a byproduct of the solution submitted by the proposer. First draw an equilateral hexagon $A'B'CD'E'F$ with opposite sides parallel ; after having drawn the first three equal sides, the entire hexagon is completely determined because of its central symmetry. Now one can adjust the figure by choosing A and B on the lines $A'F$ and $B'C$, respectively, so that $AB||A'B'$. Then by placing D on CD' and E on FE' so that DE is parallel to $D'E'$ and its distance to $D'E'$ equals the distance between AB and $A'B'$ (which are also parallel to it). It is easy to verify that the resulting figure has opposite sides parallel and the sum of their lengths constant.

3908. *Proposed by George Apostolopoulos.*

Prove that

$$\frac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$$

for each integer $n \geq 3$.

We received 21 correct submissions, with two from one solver. Below we present four different solutions.

Solution 1, by Peter Y. Woo.

Recall Bernoulli's inequality, $(1+x)^t > 1 + tx$ when $x > -1$, $x \neq 0$ and $t \geq 1$. For $m > 1$, we have

$$\begin{aligned}\left(\frac{m+1}{m}\right)^{m+1}\left(\frac{m-1}{m}\right)^{m-1} &= \left(1+\frac{1}{m}\right)^2\left(1-\frac{1}{m^2}\right)^{m-1} \\ &> \left(1+\frac{1}{m}\right)^2\left(1-\frac{m-1}{m^2}\right) \\ &= \left(\frac{m^3+1}{m^3}\right)\left(\frac{m+1}{m}\right) > 1.\end{aligned}$$

Hence $(m+1)^{m+1} > m^{2m}(m-1)^{-(m-1)}$. Setting $m = n-1$ yields the desired result.

Solution 2, by M. Benito, Ó. Ciaurri, E. Fernández and L. Roncal.

We strengthen the inequality to

$$\frac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n \left(\frac{n^2-1}{n^2}\right)^{n-1}.$$

From Bernoulli's inequality $(1+x)^r < 1 + rx$ for $x > -1$, $x \neq 0$ and $0 < r < 1$, we obtain that

$$\left(\frac{n-2}{n}\right)^{\frac{1}{n-1}} = \left(1 - \frac{2}{n}\right)^{\frac{1}{n-1}} < 1 - \frac{2}{n(n-1)} = \frac{(n+1)(n-2)}{n(n-1)}.$$

The result follows.

Solution 3, by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly); Phil McCartney; and Digby Smith, independently.

The function $f(x) = x \ln x$ (whose second derivative $1/x$ is positive for $x > 0$) is strictly convex on $(0, \infty)$. Hence, for $x > 2$,

$$2(x-1)\ln(x-1) = 2f(x-1) < f(x-2) + f(x) = (x-2)\ln(x-2) + x \ln x,$$

from which

$$(x-1)^{2x-2} < (x-2)^{x-2} x^x$$

as desired.

Solution 4, by Haohao Wang and Jerzy Wojdylo (jointly); and Angel Plaza, independently.

The function $(1+1/x)^x$ is increasing for $x > 0$. (The derivative of its logarithm is equal to $\int_x^{x+1} (t^{-1} - (x+1)^{-1}) dt$.) For $n > 2$, we have that

$$\frac{\left(1 + \frac{1}{n-2}\right)^{n-2}}{\left(1 + \frac{1}{n-1}\right)^{n-1}} < 1 < \frac{n}{n-1},$$

which yields the desired result.

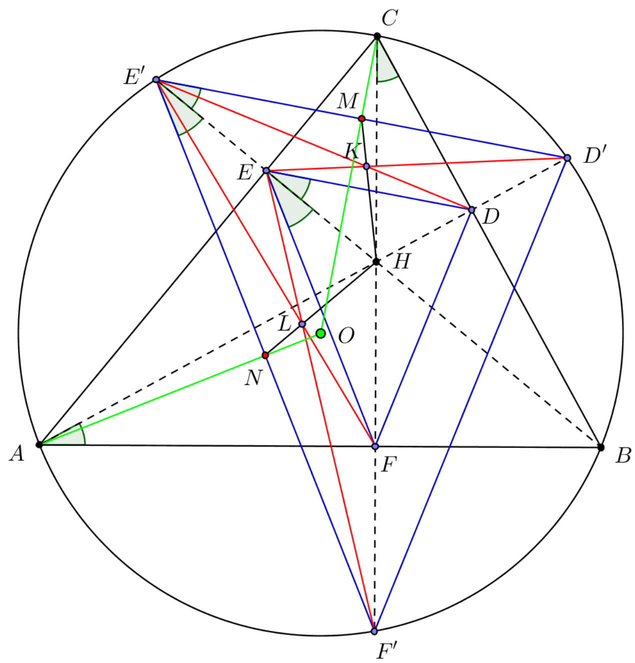
3909. *Modified proposal of Victor Oxman, Moshe Stupel and Avi Sigler.*

Given an acute-angled triangle together with its circumcircle and orthocentre, construct, with straightedge alone, its circumcentre.

Editor's Comment. The Poncelet-Steiner Theorem (1833) states that whatever can be constructed by straightedge and compass together can be constructed by straightedge alone, given a circle and its centre; but Steiner showed that given only the circle and a straightedge, the centre cannot be found. (This shows that the orthocentre must be given in the present problem; it cannot be constructed with the straightedge and circumcircle!) Details can be found in texts such as A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), or on the internet by googling the Poncelet-Steiner Theorem.

We received five correct submissions from which we present two.

Solution 1 by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.



Let H be the orthocentre of the given triangle ABC. Successively, we draw with the straightedge (as in the figure) the altitude AH, meeting the side BC at point D and the circumcircle, call it Γ, at point D'; the altitude BH, meeting the side AC at E and Γ at E'; the altitude CH, meeting the side AB at F and Γ at F'.

Draw the triangles DEF (the orthic triangle of ΔABC) and $D'E'F'$. These triangles are homothetic with respect to their common incentre H. Specifically, quadrilaterals $DEE'D'$ and $EFF'E'$ are trapezoids whose nonparallel sides meet at H. Draw the diagonals of each of these trapezoids : the lines DE' and $D'E$ meeting at the point K, and the lines EF' and $E'F$ meeting at the point L. Draw the line HK that meets $D'E'$ at M, and line HL that meets $E'F'$ at N. We know M is the midpoint of the segment $D'E'$, and N is the midpoint of the segment $E'F'$.

On the other hand, it is easily seen (the line $D'H$ bisects $\angle F'D'E'$, etc.) that the vertex C is the midpoint of one arc $D'E'$ of circumcircle $D'E'F'$ and, analogously, that the vertex A is the midpoint of arc $E'F'$. Then, the lines CM and AN are, respectively, the perpendicular bisectors of the segments $D'E'$ and $E'F'$. Consequently, the intersection point O of CM and AN is the circumcentre of $\Delta D'E'F'$ and the required circumcentre of ΔABC.

Solution 2 by Michel Bataille, Rouen, France.

The following construction is valid for all triangles ABC that are not right-angled. Without loss of generality, we suppose that the largest angle of the triangle is $\angle BAC$, and we denote the orthocentre by H and the circumcircle by Γ. Let the line AH intersect BC at D and Γ again at D'. Then D is between B and C and is the midpoint of HD'. Given a line segment with its midpoint, we can construct the parallel ℓ_C to HD' through C and the parallel ℓ_B to HD' through B. Let ℓ_C and ℓ_B intersect again Γ at C' and B', respectively. Note that $C' \neq C$ and $B' \neq B$ since BC is not a diameter of Γ. Since $\angle B'BC = \angle C'CB = 90^\circ$, BC' and $B'C$ are diameters of Γ. Their point of intersection is the desired centre O of Γ.

Editor's comments. Most of the submitted solutions were based on the theorem that says,

> Given a line segment XY and a line parallel to it, we can locate (with straightedge alone) the midpoint of XY; conversely, given a line segment XY with its midpoint M and a point P not on the line XY, we can draw (with straightedge alone) the line parallel to XY through P.

The proof rests upon the fact (as seen in the first solution above) that the line through the point of intersection of the diagonals of a trapezoid (trapezium in British English) and the point of intersection of its nonparallel sides bisects its parallel sides. Details can be found on pages 51-52 of A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, Pergamon Press, 1961.

3910. *Proposed by Paul Yiu.*

Two triangles ABC and $A'B'C'$ are homothetic. Show that if B' and C' are on the perpendicular bisectors of CA and AB respectively, then A' is on the perpendicular bisector of BC, and the homothetic center is a point on the Euler line of ABC.

We received four correct submissions. We present the solution of M. Bello, Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncall.

Let O and H be the circumcentre and orthocentre, respectively, of ΔABC. The lines $C'O$ and $B'O$ are altitudes of $A'B'C'$ (since, for example, $C'O$ is perpendicular to AB, which is parallel to its corresponding homothetic line $A'B'$). Hence, the point O is the orthocentre of $A'B'C'$. Thus, the line $A'O$ is the third altitude of $A'B'C'$ and, consequently, is the unique line through O that is perpendicular to $B'C'$ and (as $B'C'$ is parallel to BC), to BC. Consequently, $A'O$ is the perpendicular bisector of BC, as desired.

For the second claim, recall that the circumcentre O of ΔABC is the orthocenter of $\Delta A'B'C'$, and must therefore be the image of H under the dilatation that takes the first triangle to the second. In other words, the homothetic center must lie on the line OH. But OH is the Euler line of ΔABC (if it exists), and we are done.