OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(1), p. 9–10.

OC151. Let ABC be a triangle. The tangent at A to the circumcircle intersects the line BC at P. Let Q, R be the symmetrical of P with respect to the lines AB respectively AC. Prove that $BC \perp QR$.

Originally question 1 from the Japan Mathematical Olympiad.

We received five solutions. We give the solution of Michel Bataille.

We shall denote by $\angle(m, n)$ the directed angle from line m to line n (measured modulo π).

We have $\angle(PQ, PR) = \angle(AB, AC)$ (since $PQ \perp AB$ and $PR \perp AC$) and because A is the circumcentre of ΔPQR (note that $AQ = AP = AR$), we also have $\angle(PQ, PR) = \angle(\ell, AR) = \angle(AQ, \ell)$ where ℓ is the perpendicular bisector of QR. It follows that

$$\angle(AB, AC) = \angle(AQ, \ell) \tag{1}$$

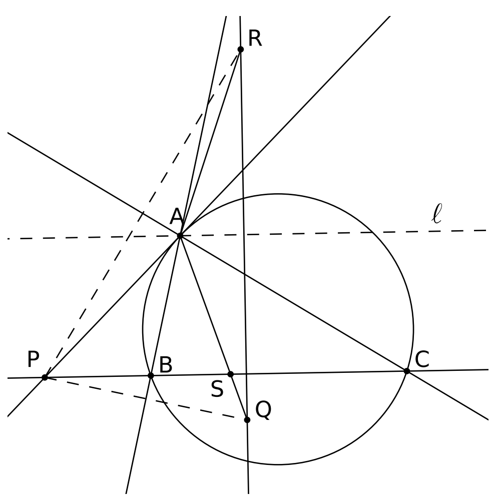
Because AP is tangent at P to the circumcircle of ΔABC, we have $\angle(AP, AB) = \angle(CA, CB)$. Therefore,

$$\angle(AB, AQ) = \angle(CA, CB)$$

and

$$\angle(AQ, BC) = \angle(AQ, AB) + \angle(BA, BC) = \angle(CB, CA) + \angle(BA, BC) = \angle(AB, AC).$$

Thus, AQ is not parallel to BC and if AQ intersects BC at S, we have $\angle(SA, SB) = \angle(AQ, BC) = \angle(AB, AC)$, hence $\angle(SA, SB) = \angle(SA, \ell)$ (by (1)). As a result, $\angle(\ell, SB) = 0$ and ℓ is parallel to BC. Since ℓ is perpendicular to QR, we conclude that BC is perpendicular to QR.



OC152. Find all non-constant polynomials $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integer coefficients whose roots are exactly the numbers $a_0, a_1, \ldots, a_{n-1}$ with the same multiplicity.

Originally question 3 from day 2 of the France TST 2012.

We received three solutions all of which assumed that the coefficients (except for possibly the first) needed to be distinct. The editor believes that the question, while possibly ambiguous, understands that repetition of coefficients is allowed and that the multiplicity condition means that each root is not repeated unless multiple coefficients are the same. As a result, the editor will give a full solution allowing repetition.

Suppose that we have such a polynomial. Write

$$P(x) = \prod_{i=0}^{n-1}(x - a_i).$$

Then, via Vieta's formulas, we have that $a_0a_1...a_{n-1} = (-1)^n a_0$. If $a_0 = a_1 = a_2 = ... = a_{k-1} = 0$ for $1 \leq k < n$, then divide out by the largest power of k and look at the remaining polynomial. This will introduce a factor of x^k. Thus, without loss of generality, we suppose that $a_0 \neq 0$. From here, clearly $n > 1$ since for $x + a_0$, we have root $-a_0$ and a_0 and $-a_0$ are distinct when $a_0 \neq 0$. So suppose $n \geq 2$. Then comparing constant terms again and cancelling the a_0 term gives $a_1a_2...a_{n-1} = (-1)^n$. Hence each root is either $a_0 \neq 0$ or ± 1 (we will include the factors of x^k at the end). Rewrite $P(x)$ as

$$P(x) = (x-1)^\ell (x+1)^m (x - a_0).$$

for integers ℓ, m. Using a binomial expansion, we see that

$$\begin{aligned} P(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \\ &= x^{\ell+m+1} + (m - \ell - a_0)x^{\ell+m} + \cdots \\ &\quad + (a_0(\ell(-1)^{\ell-1} + m(-1)^\ell) + (-1)^\ell)x + a_0(-1)^{\ell+1}. \end{aligned}$$

Comparing coefficients of the constant term yields that ℓ is odd. Hence, we see that 1 is a root of the polynomial and thus

$$0 = P(1) = 1 + \sum_{i=0}^{n-1} a_i.$$

Let S be the summation above. Squaring both sides and expanding yields

$$0 = (1+S)^2 = 1 + 2S + S^2 = 2(S+1) - 1 + S^2 = -1 + S^2.$$

Further, since $a_i^2 = 1$ when $0 < i < n$, we have via Vieta's formulas,

$$S^2 = \left(\sum_{i=0}^{n-1} a_i\right)^2 = \sum_{i=0}^{n-1} a_i^2 + 2\sum_{0 \leq i < j < n} a_ia_j = (n-1) + a_0^2 + 2a_{n-2}.$$

Combining shows that
$$n = 2 - a_0^2 - 2a_{n-2}.$$
Suppose now that $n = 2$, then $a_{n-2} = a_{2-2} = a_0$. Substituting above shows that $0 = a_0(a_0 + 2)$. Hence $a_0 = -2$. By Vieta again in this case, we see that $a_0 + a_1 = -a_1$ and so $a_1 = 1$. This gives the polynomial
$$P(x) = x^2 + x - 2.$$
Now, if $n \geq 3$, then $a_{n-2} \neq a_0$. Thus, $a_{n-2} \in \{\pm 1\}$. Hence recalling that $n = 2 - a_0^2 - 2a_{n-2}$, we see that $a_{n-2} = -1$ and $a_0 = \pm 1$ (recall $a_0 \neq 0$ and $n > 0$). This gives $n = 3$. Thus, $a_1 = -1$ and solving (from Vieta once more) $a_0 + a_1 + a_2 = -a_2$ and $a_0 a_1 a_2 = -a_0$, we see that $a_2 = 1$ and $a_0 = -1$. This gives
$$P(x) = x^3 - x^2 - x + 1.$$
Therefore, combining with the zeroes we excluded, we get the following possible solutions (for suitable n):

1. $P(x) = x^n$
2. $P(x) = x^{n-2}(x^2 + x - 2)$
3. $P(x) = x^{n-3}(x^3 - x^2 - x + 1)$

completing the proof.

OC153. Find all non-decreasing functions from real numbers to itself such that for all real numbers x, y we have
$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$
Originally question 3 from day 1 of the Turkish National Olympiad Second Round 2012.

We received two correct submissions. We give the solution of Michel Bataille.

The identity function $x \mapsto x$ is obviously a solution. We show that there are no other solutions. To this end, we consider an arbitrary solution f and denote by $E(x, y)$ the equality $f(f(x^2) + y + f(y)) = x^2 + 2f(y)$.

First, we show that $f(0) = 0$. Let $a = f(0)$. From $E(0, 0)$, we have $f(2a) = 2a$ and $E(0, 2a)$ then yields $f(5a) = 4a$. It follows that $a \leq 0$ since otherwise we would have $f(3a) = 4a$ (from $E(\sqrt{2a}, 0)$) and then $E(\sqrt{3a}, 0)$ leads to $f(5a) = 5a$, in contradiction with $f(5a) = 4a$. Now, from $E(\sqrt{-2a}, 0)$, we obtain $f(a+f(-2a)) = 0$. But, from $E(0, y)$ we deduce that $f(y) = 0$ implies $f(a + y) = 0$ and iterating, $f(2a + y) = 0$ and $f(3a + y) = 0$. Thus, we have $f(4a + f(-2a)) = 0$. However, we also have $f(4a + f(-2a)) = 2a$ (by $E(\sqrt{-2a}, 2a)$) and so $a = 0$.

Let us show that $f(z) = z$ whenever $z > 0$. Since $f(0) = 0$, we have $f(f(x^2)) = x^2$ for all x (by $E(x, 0)$) and so $f(x^2) = x^2$ because $f(x^2) < x^2$ implies $f(f(x^2)) \leq$

$f(x^2)$, that is, $x^2 \leq f(x^2)$, a contradiction. Similarly $f(x^2) > x^2$ leads to a contradiction.

Consider now $y \in (-\infty, 0)$. The relation $E(x,y)$ now writes as $f(x^2 + y + f(y)) = x^2 + 2f(y)$ and in particular $E(\sqrt{-y}, y)$ gives $f(f(y)) = -y + 2f(y)$. Note that because f is increasing, $f(y) \leq 0$ and $2f(y) - y = f(f(y)) \leq 0$, hence $3f(y) - y \leq 0$.

We distinguish the cases $y - f(y) \geq 0$ and $y - f(y) < 0$. In the former case, $f(y - f(y)) = y - f(y)$ since $y - f(y) \geq 0$ and $f(y - f(y)) = 0$ by $E(\sqrt{-2f(y)}, y)$, hence $f(y) = y$. On the other hand, if $y - f(y) < 0$, since $3f(y) - y \leq 0$, we can apply $E(\sqrt{y - 3f(y)}, f(y))$ and we obtain $0 = f(y) - y$. In both cases, $f(y) = y$.

We may conclude that $f(x) = x$ for all x, negative or not, and we are done.

OC154. For $n \in \mathbb{Z}^+$ we denote

$$x_n := \binom{2n}{n}.$$

Prove there exist infinitely many finite sets A, B of positive integers, such that $A \cap B = \emptyset$, and

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012\,.$$

Originally question 3 from day 1 of the China TST 2012.

We received one correct submission by Oliver Geupel, which we present below.

For every positive integer n, we have

$$\frac{x_{n+1}}{x_n} = \frac{2(2n+1)}{n+1}.$$

and

$$\frac{x_{n+2}}{x_n} = \frac{x_{n+2}}{x_{n+1}} \cdot \frac{x_{n+1}}{x_n} = \frac{2(2n+3)}{n+2} \cdot \frac{2(2n+1)}{n+1} = \frac{4(2n+1)(2n+3)}{(n+1)(n+2)}.$$

Hence,

$$\begin{aligned} \frac{x_{n+1}}{x_n} \cdot \frac{x_{2n}}{x_{2n+2}} &= \frac{2(2n+1)}{n+1} \cdot \frac{(2n+1)(2n+2)}{4(4n+1)(4n+3)} = \frac{(2n+1)^2}{(4n+1)(4n+3)} \\ &= \frac{8n^2+8n+2}{2(16n^2+16n+3)} = \frac{x_{8n^2+8n+1}}{x_{8n^2+8n+2}}. \end{aligned}$$

Thus,

$$\frac{x_{n+1}}{x_n} \cdot \frac{x_{2n}}{x_{2n+2}} \cdot \frac{x_{8n^2+8n+2}}{x_{8n^2+8n+1}} = 1.$$

Moreover,

$$x_1 \cdot x_5 \cdot \frac{x_{252}}{x_{251}} = 2 \cdot 252 \cdot \frac{2 \cdot 503}{252} = 2012.$$

Therefore, we may put

$$A = \{1, 5, 252, n+1, 2n, 8n^2+8n+2\}, \quad B = \{251, n, 2n+2, 8n^2+8n+1\}$$

for any $n > 252$.

OC155. There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21 groups such that the number of students in each group is equal and every two students in the same group know each other.

Originally question 3 from Vietnam Team Selection Test 2012.

No submissions were received.

From *Mathematical Cartoons* by Charles Ashbacher.