FOCUS ON...

No. 15

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A Formula of Euler

Introduction

In this number, we consider the sums $S(n,m)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m$ where n is a positive integer and m a nonnegative integer. Euler found a very simple closed form of $S(n,m)$ as long as $m\leq n$:

$$S(n,m)=0 \ \text{ if } m=0,1,\dots,n-1 \ \text{ and } \ S(n,n)=n!.$$

This result is often called Euler's formula. To appreciate its power, we give a quick solution to problem 11212 posed in the *American Mathematical Monthly* in March 2006:

> Show that for an arbitrary positive integer n, $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=0$.

Just notice that

$$\begin{aligned}\binom{2n-2r}{n-1}&=\frac{1}{(n-1)!}(2n-2r)(2n-1-2r)\cdots(n+2-2r)\\&=a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+\cdots+a_1r+a_0,\end{aligned}$$

where the coefficients $a_0,a_1,\dots,a_{n-1}$ do not depend on r. Thus,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=\sum_{j=0}^{n-1}a_j\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^j=(-1)^n\sum_{j=0}^{n-1}a_jS(n,j)$$

and the result immediately follows since each $S(n,j)$ vanishes. To use the formula in its full extent, we can also calculate $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n}$; with the help of $S(n,n)=n!$, the result 2^n is readily obtained. This illustrates the polynomial version of Euler's formula: if $P(x)$ is a polynomial whose coefficients are free of k, then

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(k)=0$$

if the degree of P is less than n and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(k)=a_nn!$$

if $P(x)=a_nx^n+\cdots$ is of degree n.

We will give three elementary proofs of Euler's formula, favoring approaches that bring out connections with algebra, analysis and combinatorics. Quite a ubiquitous formula! The reader will find other proofs and links with more advanced tools (difference operator Δ, Stirling numbers, *etc.*) in the survey article [1].

First approach: polynomials and linear algebra

We introduce the polynomials

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = x(x-1),\ \ldots,\ P_n(x) = x(x-1)\cdots(x-n+1)$$

and recall that $(P_0, P_1, \ldots, P_n)$ is a basis of the linear space formed by all polynomials of degree less than or equal to n. Now, consider $Q(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$. By successive differentiations, we readily obtain

$$Q^{(m)}(x) = \sum_{k=m}^{n}(-1)^k\binom{n}{k}P_m(k)x^{k-m}$$

for $0 \le m \le n$. On the other hand, since $Q(x) = (1-x)^n$, we also have

$$Q^{(m)}(x) = (-1)^m P_m(n)(1-x)^{n-m}.$$

Comparing the two results and taking $x = 1$ yields

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}P_m(k) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}P_m(k) = Q^{(m)}(1) = 0$$

if $m < n$ and $(-1)^n P_n(n) = (-1)^n n!$ if $m = n$. Because any polynomial $P(x)$ with $\mathrm{degree}(P) = d \le n$ is a linear combination of $P_0, P_1, \ldots, P_d$, Euler's formula is derived at once.

A recourse to the polynomial version of Euler's formula can be found in solutions II and III of **3670** [2012 : 301,302]. Another example is the following identity, extracted from a problem of the St. Petersburg Contest:

> Show that if n is an integer such that $n \ge 2$ and x, y are complex numbers with $x \ne 0$, then
>
> $$\sum_{k=0}^{n-1}\binom{n-1}{k}(x+k)^{k-1}(y+n-k)^{n-k-1} = \frac{(x+y+n)^{n-1}}{x}.$$

To prove this identity, we first transform the left-hand side L into a double sum:

$$
\begin{aligned}
L &= \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^{k-1}(x+y+n-(x+k))^{n-k-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^{k-1} \sum_{j=0}^{n-k-1} (-1)^{n-k-1-j} \binom{n-k-1}{j} (x+y+n)^j (x+k)^{n-k-1-j} \\
&= \sum_{j=0}^{n-1} (x+y+n)^j \sum_{k=0}^{n-j-1} \binom{n-1}{k} \binom{n-k-1}{j} (-1)^{n-k-1-j} (x+k)^{n-2-j} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} (x+y+n)^j \sum_{k=0}^{n-j-1} \binom{n-1-j}{k} (-1)^{n-k-1-j} (x+k)^{n-2-j} \qquad (1)
\end{aligned}
$$

where we have used the equality $\binom{n-1}{k}\binom{n-k-1}{j} = \binom{n-1}{j}\binom{n-1-j}{k}$.

From Euler's formula, the inner sum in (1) is 0 except when $j = n-1$. Thus,

$$L = \binom{n-1}{n-1}(x+y+n)^{n-1}\binom{0}{0}(-1)^0(x+0)^{-1} = \frac{1}{x}(x+y+n)^{n-1}.$$

Second approach: using a Maclaurin expansion

Consider the function f defined by

$$f(x) = (e^x-1)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}e^{kx}.$$

Clearly, $S(n,m) = f^{(m)}(0)$ where $f^{(m)}$ denotes the mth derivative of f. Thus, the values of $S(n,m)$ $(m = 0,1,\dots,n)$ can be obtained from the corresponding coefficients $\frac{f^{(m)}(0)}{m!}$ of the Maclaurin expansion of f. Since $f(x) = x^n(1+\frac{x}{2}+\cdots)^n$, Euler's formula follows.

Note that this proof makes it possible to easily obtain the value of $S(n,n+s)$ if the positive integer s is small; for example, $S(n,n+1) = \frac{n(n+1)!}{2}$ since $(1+\frac{x}{2}+\cdots)^n = 1+\frac{nx}{2}+\cdots$.

A direct application of this remark is provided by the following solution to problem 824 of the *College Mathematics Journal*, proposed in March 2006:

> Prove that the value of the sum
>
> $$\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{j+k}\frac{1}{j!k!}\binom{n-1}{j-1}\binom{n-1}{k-1}(j+k)!$$
>
> is independent of n.

The given double sum is $V_n = \sum\limits_{j=1}^{n} \frac{(-1)^j}{j!}\binom{n-1}{j-1} U_j$ where

$$U_j = \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{(j+k)!}{k!} = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}(k+2)(k+3)\cdots(k+j+1).$$

Since $(k+2)(k+3)\cdots(k+j+1)$ is a polynomial in k with degree j, we obtain $U_j = 0$ if $j < n-1$, $U_{n-1} = (-1)^n(n-1)!$ and

$$\begin{aligned} U_n &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}(k^n + (2+3+\cdots+(n+1))k^{n-1}) \\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}k^n + \frac{n(n+3)}{2}\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}k^{n-1} \\ &= (-1)^n\frac{(n-1)n!}{2} + (-1)^n\frac{n(n+3)}{2}(n-1)! = (-1)^n(n+1)n!. \end{aligned}$$

As a result, for all positive integers n,

$$V_n = \frac{(-1)^{n-1}}{(n-1)!}\cdot(n-1)U_{n-1} + \frac{(-1)^n}{n!}U_n = -(n-1)+(n+1) = 2.$$

Third approach: combinatorics

If m, n are any positive integers and $[m] = \{1, 2, \ldots, m\}$ and $[n] = \{1, 2, \ldots, n\}$, we denote by $\sigma(m, n)$ the number of surjections from $[m]$ onto $[n]$. If its range is properly restricted, a mapping f from $[m]$ to $[n]$ can be seen as a surjection from $[m]$ onto a nonempty subset of $[n]$. Thus, we obtain all the mappings from $[m]$ to $[n]$ by choosing a subset A of $[n]$ with cardinality $k \neq 0$ and a surjection from $[m]$ onto A in all possible ways. Since the total number of mappings from $[m]$ to $[n]$ is n^m, we are led to the equality

$$n^m = \sum_{k=1}^{n}\binom{n}{k}\sigma(m,k) \tag{2}$$

Now, if (a_n) and (b_n) are two sequences such that $a_n = \sum\limits_{k=0}^{n}\binom{n}{k}b_k$ for all positive integers n, then we have $b_n = \sum\limits_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_k$ for all positive integers n (a well-known inversion formula). Taking $a_0 = b_0 = 0$ and $a_n = n^m$, $b_n = \sigma(m,n)$ for $n \geq 1$ (and fixed m), equality (2) implies that $\sigma(m,n) = \sum\limits_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^m$. Euler's formula is then deduced from the fact that there are no surjections from $[m]$ onto $[n]$ if $n > m$ and that there are $n!$ such surjections (bijections actually) if $n = m$. [We have assumed that $m \geq 1$; but if $m = 0$, then $S(n, 0) = 0$ follows from $0 = (1-1)^n$ and the binomial theorem.]

Here is a related problem:

If m, n are positive integers, evaluate $S = \sum_{k=1}^{n} k\binom{n}{k}\sigma(m,k)$.

We propose the following solution. Changing the order of summation, we obtain

$$S = \sum_{k=0}^{n} k\binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^m = \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}k\binom{n}{k}\binom{k}{j}j^m.$$

Since $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{n-k}$, we see that $S = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}j^m A_j$ where

$$\begin{aligned} A_j = \sum_{k=j}^{n}(-1)^{n-k}k\binom{n-j}{n-k} &= \sum_{\ell=0}^{n-j}(-1)^{\ell}(n-\ell)\binom{n-j}{\ell} \\ &= n \cdot \sum_{\ell=0}^{n-j}(-1)^{\ell}\binom{n-j}{\ell} - \sum_{\ell=0}^{n-j}(-1)^{\ell}\cdot\ell\cdot\binom{n-j}{\ell}. \end{aligned}$$

Now, $\sum_{\ell=0}^{n-j}(-1)^{\ell}\binom{n-j}{\ell} = 0$ except for $j = n$ when the value is 1 and $\sum_{\ell=0}^{n-j}(-1)^{\ell}\ell\binom{n-j}{\ell} = 0$ except for $j = n-1$ when the value is -1. It follows that

$$S = (n^m \cdot n + (-1)\cdot n \cdot (n-1)^m) = n(n^m - (n-1)^m).$$

We conclude with two exercises.

Exercises

1. Show that for each integer $n \geq 2$

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{(-1)^{k-1}}{k}\left(1-\frac{k}{n}\right)^n = \sum_{k=1}^{n-1}\frac{1}{k+1}.$$

2. For nonnegative integer n, evaluate in closed form

$$\sum_{k=0}^{n}\frac{(-1)^k}{(k!)^2}\cdot\frac{(n+k+2)!}{(n-k)!}.$$

Reference

[1] H.W. Gould, Euler's Formula for nth Differences of Powers, *American Mathematical Monthly*, Vol. 85, June-July 1978, pp. 450-467.