THE CONTEST CORNER

No. 31

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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **March 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC134. (*Correction*). Let two tangent lines from the point $M(1,1)$ to the graph of $y = k/x$, $k < 0$ touch the graph at the points A and B. Suppose that the triangle MAB is an equilateral triangle. Find its area and the value of constant k.

CC151. Consider a non-zero integer n such that $n(n+2013)$ is a perfect square.

a) Show that n cannot be prime.

b) Find a value of n such that $n(n+2013)$ is a perfect square.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

CC153. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of positive integers is constructed as follows:

- if the last digit of a_n is less than or equal to 5, then this digit is deleted and a_{n+1} is the number consisting of the remaining digits; if a_{n+1} contains no digits, the process stops;
- otherwise, $a_{n+1} = 9a_n$.

Can one choose a_0 so that we can obtain an infinite sequence?

CC154. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ are written on the blackboard. Alice chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is only one number left on the board. What are the possible values of the final number?

CC155. Find all real solutions x to the equation $[x^2 - 2x] + 2[x] = [x]^2$. Here $[a]$ denotes the largest integer less than or equal to a.

. . . .

CC134. (*Correction*). Deux droites, issues du point $M(1, 1)$, sont tangentes à la courbe d'équation $y = k/x$ $(k < 0)$ aux points A et B. Sachant que le triangle MAB est équilatéral, déterminer la valeur de k et l'aire du triangle.

CC151. Considérer un entier naturel non-nul n tel que $n(n + 2013)$ soit un carré parfait.

a) Montrer que n ne peut pas être nombre premier.

b) Trouver une valeur de n tel que $n(n + 2013)$ soit un carré parfait.

CC152. Les cases d'un échiquier $n \times n$, avec $n \geq 5$, sont coloriées en noir ou en blanc de telle sorte que trois cases adjacentes sur une ligne, une colonne ou une diagonale ne soient pas de la même couleur. Montrer que pour tout carré 3×3 à l'intérieur de l'échiquier, deux de ses cases situées aux coins sont de couleur blanche et les deux autres sont de couleur noire.

CC153. Une suite d'entiers positifs de terme général a_n et de premier terme a_0 est définie pour tout entier naturel n de la façon suivante:

- si le dernier chiffre de a_n est inférieur ou égal a 5, alors ce chiffre est supprimé et les chiffres restants forment le terme a_{n+1}; si a_{n+1} ne contient pas de chiffre, le procédé s'arrête;
- autrement, $a_{n+1} = 9a_n$.

Peut-on choisir un entier naturel a_0 de sorte que la suite (a_n) soit infinie?

CC154. On a écrit les nombres $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ sur un tableau. Michèle en choisit deux, notés x et y, puis elle les efface et les remplace par le nombre $x+y+xy$. Elle continue ainsi jusqu'à ce qu'il ne reste plus qu'un seul nombre sur le tableau. Quelles sont les valeurs possibles de ce nombre?

CC155. Déterminer tous les nombres réels x tels que $[x^2 - 2x] + 2[x] = [x]^2$. Ici $[a]$ désigne le plus grand nombre entier inférieur ou égal à a.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(1), p. 4.

CC101. Find all pairs of whole numbers a and b such that their product ab is divisible by 175 and their sum $a+b$ is equal to 175.

Originally 7th secondary Mathematics Olympiad (Poland), 2nd level, question 1.

We received seven correct solutions. Below is the one by S. Muralidharan.

Since $b = 175 - a$ and $175 = 5^2 \cdot 7$, we need

$$\begin{aligned} a(175-a) \equiv 0 \pmod{175} &\iff a^2 \equiv 0 \pmod{175} \\ &\iff a \equiv 0 \pmod 5 \text{ and } a \equiv 0 \pmod 7. \end{aligned}$$

Thus $a \in \{0, 35, 70, 105, 140, 175\}$. Along with $b = 175 - a$, each of these values gives a solution.

CC102. In pentagon $ABCDE$, angles B and D are right. Prove that the perimeter of triangle ACE is at least $2BD$.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 5.

We received two correct submissions with similar solutions, which we present below.

Let M be the midpoint of AC and N the midpoint of EC. Since ABC and CDE are right-angled triangles, we have $MA = MB = MC$ and $NC = ND = NE$. By applying the triangle inequality twice, we obtain the inequality

$$BD \le BM + MN + ND$$

Notice that $2BM = AC, 2MN = EA$ and $2DN = EC$, so mutiplying the inequality by 2 and substituting gives the desired inequality of

$$2BD \le AC + CE + EC.$$

CC103. Let a and b be two rational numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{ab}$ is also rational. Prove that $\sqrt{a}$ and $\sqrt{b}$ must also be rationals.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 6.

There were eight solutions to this problem. We present two solutions.

Solution 1, by Šefket Arslanagić, slightly expanded by the editor.

If $a = b = 0$, then the statement holds. Otherwise it is not hard to see that a and b cannot be negative. Let r be the rational number with $r = \sqrt{a} + \sqrt{b} + \sqrt{ab}$ and note that r is positive. Then

$$\begin{aligned} &\sqrt{a} + \sqrt{ab} = r - \sqrt{b} \\ \Rightarrow\ & a + 2a\sqrt{b} + ab = r^2 - 2r\sqrt{b} + b \\ \Rightarrow\ & 2(a+r)\sqrt{b} = r^2 + b - a - ab \end{aligned}$$

Since $a + r > 0$,

$$\sqrt{b} = \frac{r^2 + b - a - ab}{2(a+r)}$$

is a rational number. Similarly it can be shown that $\sqrt{a}$ is rational.

Solution 2, a combination of the solutions by Ángel Plaza and Daniel Văcaru.

For $a = 1$, $b = 1$, or $a = b$, it is easy to see that the statement holds. Suppose otherwise. Since $\sqrt{a} + \sqrt{b} + \sqrt{ab} \in \mathbb{Q}$, its square $a + b + ab + 2\sqrt{ab} + 2a\sqrt{b} + 2b\sqrt{a}$ is rational as well. Therefore $\sqrt{ab} + a\sqrt{b} + b\sqrt{a} \in \mathbb{Q}$. Taking the difference with $\sqrt{a} + \sqrt{b} + \sqrt{ab}$, we obtain $(a-1)\sqrt{b} + (b-1)\sqrt{a} \in \mathbb{Q}$. By taking the square, we can conclude that $\sqrt{ab}$ is rational and therefore

$$\sqrt{a} + \sqrt{b} \in \mathbb{Q}. \tag{1}$$

Then

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a} + \sqrt{b}} \in \mathbb{Q}. \tag{2}$$

Adding (1) and (2) yields $\sqrt{a} \in \mathbb{Q}$, while subtracting (2) from (1) yields $\sqrt{b} \in \mathbb{Q}$.

CC104. Compare the area of an incircle of a square to the area of its circumcircle.

Question originally proposed by the editor.

We received nine correct solutions. We present the solution of Fernando Ballesta Yagüe.

Given a square of side ℓ, the radius of the incircle is r and the radius of the circumcircle is R. As r is the perpendicular from the centre to one side, it is half a side, that is, $\frac{\ell}{2}$. As the radius of the circumcircle is the distance from the centre to one vertex, it is half a diagonal, that is, $\frac{\sqrt{2}\ell}{2}$ (since the diagonal is $\sqrt{2}\ell$). Therefore, the area of the incircle is $\pi \cdot r^2 = \pi \cdot (\frac{\ell}{2})^2 = \frac{\pi\ell^2}{4}$, and the area of the circumcircle is $\pi \cdot R^2 = \pi \cdot (\frac{\sqrt{2}\ell}{2})^2 = \frac{\pi\ell^2}{2}$.

As $\frac{\pi\ell^2}{2} = 2\frac{\pi\ell^2}{4}$, the area of the circumcircle is twice the area of the incircle.

CC105. Knowing that $3.3025 < \log_{10} 2007 < 3.3026$, determine the left-most digit of the decimal expansion of 2007^{1000}.

Originally from 2007 AMQ Cegep contest.

We received five correct solutions. We present the solution of S. Muralidharan.

Given that
$$3.3025 < \log_{10} 2007 < 3.3026,$$
we have
$$3302.5 < 1000 \log_{10} 2007 = \log_{10} 2007^{1000} < 3302.6.$$
Hence
$$10^{3302} 10^{0.5} < 2007^{1000} < 10^{3302} 10^{0.6}$$
or
$$10^{3302} 10^{\frac{3}{6}} < 2007^{1000} < 10^{3302} 10^{\frac{3}{5}}.$$
Now, $10^{\frac{3}{6}} = 1000^{\frac{1}{6}}$ and since $3^6 < 1000 < 4^6$, it follows that
$$3 < 10^{\frac{3}{6}} < 4.$$
Also, $10^{\frac{3}{5}} = 1000^{\frac{1}{5}}$ and since $3^5 < 1000 < 4^5$, it follows that
$$3 < 10^{\frac{3}{5}} < 4.$$
Hence
$$3 \times 10^{3302} < 10^{\frac{3}{6}} \times 10^{3302} < 2007^{1000} < 10^{\frac{3}{5}} \times 10^{3302} < 4 \times 10^{3302}.$$
Therefore, the left most decimal digit of 2007^{1000} is 3.