CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(1), p. 4.

CC101. Find all pairs of whole numbers a and b such that their product ab is divisible by 175 and their sum $a+b$ is equal to 175.

Originally 7th secondary Mathematics Olympiad (Poland), 2nd level, question 1.

We received seven correct solutions. Below is the one by S. Muralidharan.

Since $b = 175 - a$ and $175 = 5^2 \cdot 7$, we need

$$\begin{aligned} a(175-a) \equiv 0 \pmod{175} &\iff a^2 \equiv 0 \pmod{175} \\ &\iff a \equiv 0 \pmod 5 \text{ and } a \equiv 0 \pmod 7. \end{aligned}$$

Thus $a \in \{0, 35, 70, 105, 140, 175\}$. Along with $b = 175 - a$, each of these values gives a solution.

CC102. In pentagon $ABCDE$, angles B and D are right. Prove that the perimeter of triangle ACE is at least $2BD$.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 5.

We received two correct submissions with similar solutions, which we present below.

Let M be the midpoint of AC and N the midpoint of EC. Since ABC and CDE are right-angled triangles, we have $MA = MB = MC$ and $NC = ND = NE$. By applying the triangle inequality twice, we obtain the inequality

$$BD \leq BM + MN + ND$$

Notice that $2BM = AC, 2MN = EA$ and $2DN = EC$, so mutiplying the inequality by 2 and substituting gives the desired inequality of

$$2BD \leq AC + CE + EC.$$

CC103. Let a and b be two rational numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{ab}$ is also rational. Prove that $\sqrt{a}$ and $\sqrt{b}$ must also be rationals.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 6.

There were eight solutions to this problem. We present two solutions.

Solution 1, by Šefket Arslanagić, slightly expanded by the editor.

If $a = b = 0$, then the statement holds. Otherwise it is not hard to see that a and b cannot be negative. Let r be the rational number with $r = \sqrt{a} + \sqrt{b} + \sqrt{ab}$ and note that r is positive. Then

$$\begin{aligned} &\sqrt{a} + \sqrt{ab} = r - \sqrt{b} \\ \Rightarrow\, &a + 2a\sqrt{b} + ab = r^2 - 2r\sqrt{b} + b \\ \Rightarrow\, &2(a + r)\sqrt{b} = r^2 + b - a - ab \end{aligned}$$

Since $a + r > 0$,

$$\sqrt{b} = \frac{r^2 + b - a - ab}{2(a + r)}$$

is a rational number. Similarly it can be shown that $\sqrt{a}$ is rational.

Solution 2, a combination of the solutions by Ángel Plaza and Daniel Văcaru.

For $a = 1$, $b = 1$, or $a = b$, it is easy to see that the statement holds. Suppose otherwise. Since $\sqrt{a}+\sqrt{b}+\sqrt{ab} \in \mathbb{Q}$, its square $a+b+ab+2\sqrt{ab}+2a\sqrt{b}+2b\sqrt{a}$ is rational as well. Therefore $\sqrt{ab} + a\sqrt{b} + b\sqrt{a} \in \mathbb{Q}$. Taking the difference with $\sqrt{a} + \sqrt{b} + \sqrt{ab}$, we obtain $(a - 1)\sqrt{b} + (b - 1)\sqrt{a} \in \mathbb{Q}$. By taking the square, we can conclude that $\sqrt{ab}$ is rational and therefore

$$\sqrt{a} + \sqrt{b} \in \mathbb{Q}. \tag{1}$$

Then

$$\sqrt{a} - \sqrt{b} = \frac{a - b}{\sqrt{a} + \sqrt{b}} \in \mathbb{Q}. \tag{2}$$

Adding (1) and (2) yields $\sqrt{a} \in \mathbb{Q}$, while subtracting (2) from (1) yields $\sqrt{b} \in \mathbb{Q}$.

CC104. Compare the area of an incircle of a square to the area of its circumcircle.

Question originally proposed by the editor.

We received nine correct solutions. We present the solution of Fernando Ballesta Yagüe.

Given a square of side ℓ, the radius of the incircle is r and the radius of the circumcircle is R. As r is the perpendicular from the centre to one side, it is half a side, that is, $\frac{\ell}{2}$. As the radius of the circumcircle is the distance from the centre to one vertex, it is half a diagonal, that is, $\frac{\sqrt{2}\ell}{2}$ (since the diagonal is $\sqrt{2}\ell$). Therefore, the area of the incircle is $\pi \cdot r^2 = \pi \cdot (\frac{\ell}{2})^2 = \frac{\pi\ell^2}{4}$, and the area of the circumcircle is $\pi \cdot R^2 = \pi \cdot (\frac{\sqrt{2}\ell}{2})^2 = \frac{\pi\ell^2}{2}$.

As $\frac{\pi\ell^2}{2} = 2\frac{\pi\ell^2}{4}$, the area of the circumcircle is twice the area of the incircle.

CC105. Knowing that $3.3025 < \log_{10} 2007 < 3.3026$, determine the left-most digit of the decimal expansion of 2007^{1000}.

Originally from 2007 AMQ Cegep contest.

We received five correct solutions. We present the solution of S. Muralidharan.

Given that

$$3.3025 < \log_{10} 2007 < 3.3026,$$

we have

$$3302.5 < 1000 \log_{10} 2007 = \log_{10} 2007^{1000} < 3302.6.$$

Hence

$$10^{3302} 10^{0.5} < 2007^{1000} < 10^{3302} 10^{0.6}$$

or

$$10^{3302} 10^{\frac{3}{6}} < 2007^{1000} < 10^{3302} 10^{\frac{3}{5}}.$$

Now, $10^{\frac{3}{6}} = 1000^{\frac{1}{6}}$ and since $3^6 < 1000 < 4^6$, it follows that

$$3 < 10^{\frac{3}{6}} < 4.$$

Also, $10^{\frac{3}{5}} = 1000^{\frac{1}{5}}$ and since $3^5 < 1000 < 4^5$, it follows that

$$3 < 10^{\frac{3}{5}} < 4.$$

Hence

$$3 \times 10^{3302} < 10^{\frac{3}{6}} \times 10^{3302} < 2007^{1000} < 10^{\frac{3}{5}} \times 10^{3302} < 4 \times 10^{3302}.$$

Therefore, the left most decimal digit of 2007^{1000} is 3.