SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the following authors for the solution of problem 3853: George Apostolopoulos, Phil McCartney, and Daniel Văcaru for a minor generalization of the problem. Sincere apologies for the oversight.

Statements of the problems in this section originally appear in 2013: 39(9), p. 413–417.

3881. *Proposed by Ovidiu Furdui.*

Calculate

$$\sum_{n=2}^{\infty}\left(n^2\ln\left(1-\frac{1}{n^2}\right)+1\right).$$

We received ten correct submissions. We present the solution by Michel Bataille.

Let

$$U_n=n^2\ln\left(1-\frac{1}{n^2}\right)+1.$$

It is readily checked that for all $n\geq 2$

$$U_n=1+2\ln(n-1)+(a_n-a_{n-1})+(b_n-b_{n-1})-2(c_n-c_{n-1})$$

where $a_n=\ln(n),\ b_n=n^2(\ln(n+1)-\ln(n))$ and $c_n=n\ln(n)$. It follows that for all integer $N>2$:

$$\begin{aligned}\sum_{n=2}^{N}U_n&=N-1+2\ln[(N-1)!]+a_N-a_1+b_N-b_1-2(c_N-c_1)\\&=N-1+2\ln\left(\frac{N!}{N}\right)+\ln(N)+N^2(\ln(N+1)-\ln(N))-\ln(2)-2N\ln(N)\\&=2\ln(N!)-2N\ln(N)-\ln(N)+N+N^2\ln\left(1+\frac{1}{N}\right)-1-\ln(2).\end{aligned}$$

We know that

$$\ln(N!)=N\ln(N)-N+\frac{\ln(N)}{2}+\ln(\sqrt{2\pi})+o(1),$$

and

$$\ln\left(1+\frac{1}{N}\right)=\frac{1}{N}-\frac{1}{2N^2}+o\left(\frac{1}{N^2}\right)$$

as $N\to\infty$.

Thus,

$$\sum_{n=2}^{N} U_n = 2\ln(\sqrt{2\pi}) - \frac{1}{2} - 1 - \ln(2) + o(1)$$

and we can conclude that

$$\sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \left(n^2 \ln\left(1 - \frac{1}{n^2}\right) + 1\right) = \ln(\pi) - \frac{3}{2}.$$

3882. *Originally proposed by Mehmet Sahin; corrected version by Arkady Alt.*

Let ABC be a right angle triangle with $\angle CAB = 90^\circ$ and hypotenuse a. Let $[AD]$ be an altitude and let I_1 and I_2 be the incenters of the triangles ABD and ADC, respectively. Let ρ be the radius of the circle through the points B, I_1 and I_2 and let r be the inradius of the triangle ABC. Prove that

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}$$

and $\min \dfrac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

We received seven correct solutions.

Solution to Part 1, by AN-anduud Problem Solving Group.

Let E and F be the points where the line I_1I_2 meets the sides AB and AC, respectively. From the given conditions, both triangles I_1BD and I_2AD have angles of 45° and $\frac{B}{2}$, so they are similar and

$$k := \frac{DI_1}{DB} = \frac{DI_2}{DA}.$$

It follows that the dilative rotation defined by a rotation through -45° about D followed by the dilatation centred at D with ratio k takes B to I_1 and A to I_2, and therefore, it takes BA to I_1I_2. From this we deduce that $\angle AEF = \angle AFE = 45^\circ$, whence

$$\begin{aligned} \angle BI_1I_2 + \angle I_2CB &= \left(\frac{1}{2}\angle ABC + 135^\circ\right) + \frac{1}{2}\angle ACB \\ &= 135^\circ + \frac{1}{2}(\angle ABC + \angle ACB) = 180^\circ. \end{aligned}$$

Therefore, I_1BCI_2 is a cyclic quadrilateral with circumradius ρ. If we denote the centre of the circumcircle by O' (and recall that the angle at O' equals twice any inscribed angle subtended by the same chord), we have

$$\angle BO'C = \angle I_1O'C + \angle BO'I_2 - \angle I_1O'I_2 = \angle B + \angle C - 2\psi,$$

where we set $2\psi = \angle I_1O'I_2$. Thus $\angle BO'C = 90^\circ - 2\psi$, and we have

$$a^2 = \rho^2 + \rho^2 - 2\rho^2\cos(90^\circ - 2\psi) = 2\rho^2(1 - \sin 2\psi) = 2\rho^2 - 4\rho^2 \cdot \sin\psi\cos\psi.$$

As a chord of a circle whose radius is ρ, $I_1I_2 = 2\rho\sin\psi$. But it is known that $I_1I_2 = \sqrt{2}r$. Briefly, we denote the inradii of triangles DBA and DAC by r_1 and r_2, so that the similarity of these triangles to ΔABC yields $r_1 = \frac{c}{a}r$ and $r_2 = \frac{b}{a}r$. In the right triangle DI_1I_2 we have $DI_1 = r_1\sqrt{2}$ and $DI_2 = r_2\sqrt{2}$, whence

$$I_1I_2 = \sqrt{2\left(r_1^2 + r_2^2\right)} = \frac{r\sqrt{2}}{a}\sqrt{b^2 + c^2} = \sqrt{2}r.$$

Equating the two expressions for I_1I_2 gives us

$$\sin\psi = \frac{r\sqrt{2}}{2\rho} \quad \text{and} \quad \cos\psi = \sqrt{1 - \sin^2\psi} = \frac{1}{2\rho}\sqrt{4\rho^2 - 2r^2}.$$

Hence,

$$a^2 = 2\rho^2 - \sqrt{8\rho^2r^2 - 4r^4}.$$

Finally, ρ^2 will be the larger root of the resulting quadratic with $x = \rho^2$, so we calculate

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}.$$

Editor's Comment. Solver Modak derived the formula

$$\rho = \sqrt{\frac{a^2 + bc}{2}},$$

which can easily be seen to be equivalent to the requested form using $2r + a = b + c$ and $b^2 + c^2 = a^2$.

Solution to Part 2, by Prithwijit De.

Dividing ρ by r leads to

$$\frac{\rho}{r} = \sqrt{\frac{a^2 + 2ar + 2r^2}{2r^2}} = \sqrt{\frac{1}{2}\left(\frac{a}{r} + 1\right)^2 + \frac{1}{2}}.$$

Thus $\frac{\rho}{r}$ attains a minimum value when $\frac{r}{a}$ attains a maximum value. Because $\frac{b}{a} = \sin B$ and $\frac{c}{a} = \cos B$, we conclude that

$$\frac{r}{a} = \frac{b + c - a}{2a} = \frac{\sin B + \cos B - 1}{2} = \frac{\sqrt{2}\sin(B + 45^\circ) - 1}{2} \le \frac{\sqrt{2} - 1}{2};$$

equality is achieved when $b = c = \frac{a}{\sqrt{2}}$. Hence $\min\frac{a}{r} = \left(\max\frac{r}{a}\right)^{-1} = 2(\sqrt{2} + 1)$, and it follows that

$$\min\frac{\rho}{r} = \sqrt{9 + 6\sqrt{2}} = \sqrt{(\sqrt{3} + \sqrt{6})^2} = \sqrt{3} + \sqrt{6}.$$

3883. *Proposed by Max A. Alekseyev.*

Let a, b, c, d be positive integers such that $a + b$ and $ad + bc$ are odd. Prove that if $2^a - 3^b > 1$, then $2^a - 3^b$ does not divide $2^c + 3^d$.

No solution was received. We present the solution of the proposer.

Since $2^a - 3^b > 1$ and $a + b$ is odd, we have $a > 2$. Consider two cases.

Case I: If a is even, then b and c are odd.

In this case $2^a \equiv 16 \pmod{24}$ and $3^b \equiv 3 \pmod{24}$, hence $2^a - 3^b \equiv 13 \pmod{24}$. For every prime divisor p of $2^a - 3^b$, we have $3^b \equiv 2^a \pmod{p}$, implying that 3 is a square modulo p, i.e. $p \equiv 1, 11, 13,$ or $23 \pmod{24}$. Since the products of residues 1 and 11 modulo 24 cannot produce the residue 13, there exists a prime divisor $p \equiv 13$ or $23 \pmod{24}$. For such p, the number 3 (and therefore 3^d) is a quadratic residue, whereas -2 (and therefore $(-2)^c$) is not. Hence $-2^c \not\equiv 3^d \pmod{p}$. So p does not divide $2^c + 3^d$ and thus neither does $2^a - 3^b$.

Case II: If a is odd, then b is even and d odd.

This time $2^a - 3^b \equiv 23 \pmod{24}$. Following the same steps as above, we find that 2 is a square modulo p for any prime divisor p of $2^a - 3^b$. We conclude that there has to be a prime divisor $p \equiv 17$ or $23 \pmod{24}$ for which 2^c is a quadratic residue, whereas -3^d is not. Again we obtain that p (and therefore $2^a - 3^b$) does not divide $2^c + 3^d$.

3884. *Proposed by Mihai Bogdan.*

Let a, b, c and d be positive real numbers such that $a + b + c + d = k$, where $k \in (0, 8)$. Prove that:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \geq \frac{k(8-k)}{8}.$$

When does the equality hold?

We received ten correct submissions. We present two different solutions.

Solution 1, by AN-anduud Probem Solving Group and Šefket Arslanagić, done independently.

Applying the AM-GM inequality twice, we have:

$$\begin{aligned}
&\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \\
&\qquad = \sum \left(a - \frac{ab^2}{b^2+1}\right) \geq \sum \left(a - \frac{ab^2}{2b}\right) = \sum a - \frac{1}{2}\sum ab \\
&\qquad = k - \frac{1}{2}(ab + bc + cd + da) = k - \frac{1}{2}(a+c)(b+d) \\
&\qquad \geq k - \frac{1}{2}\left(\frac{a+b+c+d}{2}\right)^2 = k - \frac{1}{8}k^2 = \frac{k(8-k)}{8}.
\end{aligned}$$

Equality holds if and only if k=4 and $a = b = c = d = 1$.

Solution 2, by Cao Minh Quang and Oliver Geupel, done independently.

As in Solution 1 above, we have:

$$\begin{aligned}\sum \frac{a}{b^2+1} &\geq \sum a - \frac{1}{2}\sum ab \\ &= k - \frac{1}{8}((a+b+c+d)^2 - (a-b+c-d)^2) \\ &\geq k - \frac{1}{8}(a+b+c+d)^2 = k - \frac{1}{8}k^2 = \frac{k(8-k)}{8}.\end{aligned}$$

Equality holds if and only if k=4 and $a = b = c = d = 1$.

3885. *Proposed by Oai Thanh Dao.*

Let ABC be a triangle and let F be a point that lies on the circumcircle of ABC. Further, let H_a, H_b and H_c denote projections of the orthocenter H onto sides BC, AC and AB, respectively. The three circles AH_aF, BH_bF and CH_cF meet the three sides BC, AC and AB at points A_1, B_1 and C_1, respectively. Prove that the points A_1, B_1 and C_1 are collinear.

We received ten correct submissions. We present two solutions.

Solution 1, by Titu Zvonaru modified by the editor.

Because the points A, H_a, A_1, F lie on a circle and $\angle AH_aA_1 = 90^\circ$, AA_1 must be a diameter of the circle and $\angle AFA_1$ must also equal 90°. Moreover, if we denote by A' the point where FA_1 again meets the circumcircle of ΔABC, AA' must be a diameter of that circle. We can reverse the argument and define AA' to be a diameter of the circumcircle, and then define A_1 to be the point where $A'F$ intersects the line BC. Similarly, define B' and C' to be diametrically opposite B and C so that

$$A_1 = A'F \cap BC, \quad B_1 = B'F \cap CA, \quad \text{and} \quad C_1 = C'F \cap AB.$$

Applying Pascal's theorem to the cyclic hexagon $AA'FB'BC$ we find that

$$AA' \cap B'B = O, \quad A'F \cap BC = A_1, \quad \text{and} \quad FB' \cap CA = B_1$$

are collinear. Similarly, applying the theorem to the cyclic hexagon $BB'FC'CA$, we deduce that O, B_1, C_1 are also collinear. We conclude that A_1, B_1, and C_1 all lie on a line through O.

Solution 2, by Michel Bataille.

With H the orthocentre, let the line FH meet the circumcircle Γ of ΔABC again at F_1, and let F' be the midpoint of HF_1. Expressing half the power of H with respect to Γ in four ways, we obtain

$$HF \cdot HF' = HA \cdot HH_a = HB \cdot HH_b = HC \cdot HH_c.$$

(Here and in what follows, all distances are signed.) It follows that F' is on the circles AH_aF, BH_bF and CH_cF and, therefore, the respective centres O_a, O_b, O_c of these circles are collinear (on the perpendicular bisector of FF'). Since O_a is the midpoint of AA_1 and A_1 is on BC, the line VW joining the midpoints V and W of CA and AB passes through O_a. The homothety with centre A and factor 2 maps V to C, W to B and O_a to A_1. As a result, we have

$$\frac{A_1C}{BA_1} = \frac{O_aV}{WO_a}.$$

In a similar way, if U is the midpoint of BC, we obtain $\frac{B_1A}{CB_1} = \frac{O_bW}{UO_b}$ and $\frac{C_1B}{AC_1} = \frac{O_cU}{VO_c}$ and so

$$\frac{A_1C}{BA_1} \cdot \frac{B_1A}{CB_1} \cdot \frac{C_1B}{AC_1} = \frac{O_aV}{WO_a} \cdot \frac{O_bW}{UO_b} \cdot \frac{O_cU}{VO_c}.$$

The collinearity of A_1, B_1, C_1 immediately results from the collinearity of O_a, O_b, O_c with the help of Menelaus's theorem and its converse.

Editor's Comments. Bataille pointed out that some care must be taken in the definitions of the points A_1, B_1, C_1: when F equals a vertex of ΔABC or when it is a reflection of H in a side of the triangle, one or more of the circles $AH_aF, \ldots,$ will not be defined. If one allows points at infinity, the alternative definition of the points A_1, B_1, C_1 used in Solution 1 is valid for all positions of F on the circumcircle. De observed that the Miquel point common to the circles AB_1C_1, A_1BC_1, and A_1B_1C lies on the circumcircle of ΔABC, and F is its image under reflection in the line $A_1B_1C_1$. Several readers used elegant arguments to set up the converse of Menelaus's theorem, but failed to prove that one or three of A_1, B_1, C_1 are exterior to ΔABC. (This editor also failed in his attempts to devise a convincing proof. Recall that should zero or two of the points be exterior, then Ceva's theorem, not Menelaus's, would apply.) Note that Bataille's argument in Solution 2 cleverly circumvents the difficulty.

3886. *Proposed by Michel Bataille.*

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number and let $H_0 = 0$. Prove that for $n \geq 1$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

We received eight correct submissions. We present the solution by José H. Nieto.

Let

$$f(n) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k.$$

Since $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we have

$$\begin{aligned} f(n) &= \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} 2^k H_k + \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k} 2^k H_k \\ &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} 2^{k+1} \left(H_k + \frac{1}{k+1} \right) - f(n-1) \\ &= 2f(n-1) + \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \frac{2^{k+1}}{k+1} - f(n-1) \\ &= f(n-1) + \frac{1}{n} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k \\ &= f(n-1) + \frac{1}{n} \left((2-1)^n - (-1)^n \right) \\ &= f(n-1) + \frac{1}{n} \left(1 - (-1)^n \right). \end{aligned}$$

Since we may write

$$f(n) = f(1) + \sum_{k=2}^{n} (f(k) - f(k-1)),$$

we have

$$\begin{aligned} f(n) &= 2 + \sum_{k=2}^{n} \frac{1}{k} \left(1 - (-1)^k \right) = \sum_{k=1}^{n} \frac{1}{k} \left(1 - (-1)^k \right) = \sum_{\substack{1 \le k \le n \\ k \text{ odd}}} \frac{2}{k} \\ &= 2H_n - \sum_{\substack{1 \le k \le n \\ k \text{ even}}} \frac{2}{k} = 2H_n - H_{\lfloor n/2 \rfloor}, \end{aligned}$$

as required.

3887. *Proposed by Dao Hoang Viet.*

Let a, b and c be positive real numbers. Prove that

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \ge \frac{9}{(a+b+c)^2}.$$

We received 16 correct submissions. We present three solutions.

Solution 1, by Arkady Alt and Dragoljub Milošević, done independently.

Since $a^2 + ab + b^2 \ge 3ab$, we have

$$\frac{a^3}{a^2+ab+b^2} = a - \frac{a^2b+ab^2}{a^2+ab+b^2} \ge a - \frac{ab(a+b)}{3ab} = a - \frac{a+b}{3} = \frac{2a-b}{3}.$$

Similarly,

$$\frac{b^3}{b^2+bc+c^2} \geq \frac{2b-c}{3} \quad \text{and} \quad \frac{c^3}{c^2+ca+a^2} \geq \frac{2c-a}{3}.$$

Adding up these three inequalities yields

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3}.$$

Hence, by the AM-GM inequality, we have

$$\begin{aligned}\sum_{cyc} \frac{a^2}{bc(a^2+ab+b^2)} &= \frac{1}{abc}\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \\ &\geq \frac{a+b+c}{3abc} = \frac{(a+b+c)^3}{3abc(a+b+c)^2} \\ &\geq \frac{27abc}{3abc(a+b+c)^2} = \frac{9}{(a+b+c)^2}.\end{aligned}$$

Equality occurs if and only if $a=b=c$.

Solution 2, by AN-anduud Problem Solving Group.

Applying the AM-GM inequality twice, we get:

$$\begin{aligned}\sum_{cyc} \frac{a^2}{bc(a^2+ab+b^2)} &= \sum_{cyc} \frac{(a^2+ab+b^2)-b(a+b)}{bc(a^2+ab+b^2)} \\ &= \sum_{cyc} \frac{1}{bc} - \sum_{cycl} \frac{a+b}{c(a^2+ab+b^2)} \\ &= \frac{a+b+c}{abc} - \sum_{cycl} \frac{a+b}{c(a^2+ab+b^2)} \\ &\geq \frac{a+b+c}{abc} - \sum_{cyc} \frac{a+b}{3c\cdot\sqrt[3]{a^2\cdot ab\cdot b^2}} = \frac{1}{3}\cdot\frac{a+b+c}{abc} \\ &\geq \frac{1}{3}\cdot\frac{a+b+c}{\left(\frac{a+b+c}{3}\right)^3} = \frac{9}{(a+b+c)^2}.\end{aligned}$$

Equality occurs if and only if $a=b=c$.

Solution 3, by Titu Zvonaru.

The given inequality is equivalent to

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{9abc}{(a+b+c)^2}. \tag{1}$$

Since
$$\sum_{cyc} \frac{a^3 - b^3}{a^2 + ab + b^2} = \sum_{cyc} (a - b) = 0,$$
(1) is equivalent to
$$\sum_{cyc} \frac{a^3 + b^3}{a^2 + ab + b^2} \geq \frac{18abc}{(a + b + c)^3}. \tag{2}$$
Next,
$$\frac{a^3 + b^3}{a^2 + ab + b^2} \geq \frac{a + b}{3}$$
is equivalent, in succession, to
$$\frac{a^2 - ab + b^2}{a^2 + ab + b^2} \geq \frac{1}{3},$$
$$3(a^2 - ab + b^2) \geq a^2 + ab + b^2,$$
or
$$a^2 + b^2 \geq 2ab,$$
which is true. Hence, by the AM-GM inequality we have:
$$\begin{aligned}\sum_{cyc} \frac{a^3 + b^3}{a^2 + ab + b^2} &\geq \sum_{cyc} \frac{a + b}{3}\\ &= \frac{2}{3}(a + b + c) = \frac{2}{3} \cdot \frac{(a + b + c)^3}{(a + b + c)^2}\\ &\geq \frac{2}{3} \cdot \frac{27abc}{(a + b + c)^2} = \frac{18abc}{(a + b + c)^2},\end{aligned}$$
which establishes (2) and completes the proof.

Editor's Comment. As usual, Wagon provided a proof based on the algebraic algorithm FindInstance which took 5 minutes to confirm the result.

3888. *Proposed by Peter Woo.*

My greatly admired high school teacher taught me one foolproof method when solving triangles. Suppose in triangle ABC you are given the measure of $\angle A$ and the lengths of the adjacent sides b and c; then to find the remaining angles in terms of the given quantities, one should use the law of cosines to find the length of the third side and then the law of sines to find measures of $\angle B$ and $\angle C$. Or so I was taught. But after many years, I found a way to solve this problem while avoiding the cosine law and the use of square roots. Can you discover such a way?

There were 10 correct solutions to this problem, split into two approaches.

We feature one of each type.

Solution 1, by Oliver Geupel.

First, applying the law of tangents

$$\tan\frac{\angle B-\angle C}{2}=\frac{b-c}{b+c}\cot\frac{\angle A}{2},$$

we find $\frac{\angle B-\angle C}{2}$. Then, adding and subtracting

$$\frac{\angle B-\angle C}{2}\qquad\text{and}\qquad\frac{\angle B+\angle C}{2}=90^\circ-\frac{\angle A}{2},$$

we compute $\angle B$ and $\angle C$.

Solution 2, by Roy Barbara.

To calculate the angle B (angle C can be calculated similarly), let O denote the projection of C onto the line AB. Consider the coordinate system with origin O that contains the triangle ABC with coordinates $A(\alpha,0)$, $B(\beta,0)$, and $C(0,h)$ with $h>0$. Then $h=b\sin A$. From $\alpha=-b\cos A$ we obtain $\beta=\alpha+c=c-b\cos A$. Finally,

$$\cot B=\frac{\beta}{h}=\frac{c-b\cos A}{b\sin A}.$$

3889. *Proposed by Cristinel Mortici.*

Prove that

$$e^{\pi}>\left(\frac{e^2+\pi^2}{2e}\right)^e.$$

We received ten correct solutions, and a Mathematica verification. We present two solutions which are representative of all solutions.

Solution 1, by Paolo Perfetti.

More generally we prove that for $x\in\mathbb{R}$,

$$e^{|x|}>\left(\frac{e^2+x^2}{2e}\right)^e\quad\Longleftrightarrow\quad|x|\geq e.$$

The inequality is equivalent to

$$|x|\geq e\ln(e^2+x^2)-e\ln 2-e.$$

Let

$$f(x)=|x|-e\ln(e^2+x^2)+e\ln 2+e.$$

The function f is even, ie. $f(x)=f(-x)$, so it suffices to consider $x\geq 0$. Clearly $f(0)=e(\ln 2-1)<0$, $f(e)=0$ and

$$f'(x) = 1 - \frac{2xe}{e^2 + x^2} = \frac{(e-x)^2}{e^2+x^2} \geq 0.$$

The result follows.

Solution 2, by C.R. Pranesachar.

We shall prove the more general inequality

$$e^{e+x} > \left(\frac{e^2 + (e+x)^2}{2e}\right)^e \qquad (1)$$

for all $x > 0$. The desired inequality follows by substituting $x = \pi - e$, which is positive. To prove (1), we replace x by et, where $t > 0$. Then (1) reduces to

$$e^{e(1+t)} > \left(\frac{2e^2 + 2e^2t + e^2t^2}{2e}\right)^e,$$

for $t > 0$, which is equivalent to $e^t > 1 + t + \frac{t^2}{2}$. This is true for $t > 0$, since the right-hand side is the degree two Maclaurin expansion of e^t, and all of the terms in the Maclaurin series for e^t are positive. This proves (1). Note that (1) becomes equality for $x = 0$, and the inequality sign flips for $x < 0$, since for $t < 0$, we have $e^t < 1 + t + \frac{t^2}{2}$, which may be proven by some simple derivative arguments. We are done.

Editor's Comments. The main ideas of Solution 1, i.e. taking a logarithm, defining a function, and taking its derivative to prove the inequality, were utilized by a large majority of the solvers, though the specific function used varied. The Taylor approximation inequality in Solution 2 is also the main step in the proposer's solution, which looks slightly different and is used as the starting point, as opposed to arising at the end of the solution. Paolo Perfetti commented that this problem showed up in *Mathematical Reflections*, 2015–2, as problem U334.

3890*. *Proposed by Šefket Arslanagić.*

Let $\alpha, \beta, \gamma \in \mathbb{R}$. Prove or disprove that

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha+\beta+\gamma)| \leq 1 + \frac{3\sqrt{3}}{2}.$$

We received nine correct solutions. The proposed inequality is false in general, which was pointed out by all the solvers who gave various counterexamples as listed below. We also present a correct version with proof.

Solution 1, various counterexamples.

Let S denote the left side of the given inequality.

a) AN-anduud Problem Solving Group proved that the inequality is reversed if $\alpha=\beta=\frac{\pi}{3}+x$ and $\gamma=\frac{\pi}{3}-x$ for all $x\in(0,\frac{\pi}{3})$.

b) Norman Hodžić and Salem Malikić (jointly) gave, without proof, the counterexample $\alpha=\beta=\gamma=\frac{5\pi}{8}$, which is essentially the same as a) since $\sin\left(\frac{5\pi}{8}\right)=\sin\left(\frac{3\pi}{8}\right)$ and $\cos\left(\frac{15\pi}{8}\right)=\cos\left(\frac{\pi}{8}\right)=-\cos\left(\frac{9\pi}{8}\right)$.

c) David Manes gave the counterexample $\alpha=\beta=\gamma=\frac{7\pi}{18}$.

d) Dragoljub Milošević and Digby Smith (independently) showed that when $\alpha=\beta=\gamma=\frac{5\pi}{12}$, then $S=\frac{3\sqrt{6}+5\sqrt{2}}{4}>1+\frac{3\sqrt{3}}{2}$.

e) Roberto de la Cruz, Missouri State University Problem Solving Group and C.R. Pranesachar (independently) showed that when $\alpha=\beta=\gamma=\frac{3\pi}{8}$, then $S=2\sqrt{2+\sqrt{2}}>1+\frac{3\sqrt{3}}{2}$. Missouri State University Problem Solving Group also stated, without proof, the general result that for real numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$ we have the sharp inequality below:

$$\sum_{i=1}^{n}|\sin\alpha_i|+\left|\cos\left(\sum_{i=1}^{n}\alpha_i\right)\right|\le\begin{cases}(n+1)\cos\frac{\pi}{2(n+1)} & \text{if } n \text{ is odd,}\\ n+1 & \text{if } n \text{ is even.}\end{cases}$$

Solution 2, a correct version by Roy Barbara.

We prove that

$$|\sin\alpha|+|\sin\beta|+|\sin\gamma|+|\cos(\alpha+\beta+\gamma)|\le 2\sqrt{2+\sqrt{2}} \qquad (1)$$

with equality if and only if for some integers $l,m,n\in\mathbb{Z}$ we have

$$(\alpha,\beta,\gamma)=\left(\tfrac{3\pi}{8}+l\pi,\tfrac{3\pi}{8}+m\pi,\tfrac{3\pi}{8}+n\pi\right) \text{ or } \left(\tfrac{5\pi}{8}+l\pi,\tfrac{5\pi}{8}+m\pi,\tfrac{5\pi}{8}+n\pi\right). \qquad (2)$$

(Thus, the upper bound is sharp.)

First, consider the function $g(x)=3\sin x-\cos 3x$ and $h(x)=3\sin x+\cos 3x$ for $x\in[0,\pi]$. Using the identities $\cos\left(\frac{\pi}{8}\right)=\sin\left(\frac{3\pi}{8}\right)=\frac{1}{2}\sqrt{2+\sqrt{2}}$ and $\sin\left(\frac{\pi}{8}\right)=\cos\left(\frac{3\pi}{8}\right)=\frac{1}{2}\sqrt{2-\sqrt{2}}$, it is routine to check that $g(x)$ attains its maximum of $2\sqrt{2+\sqrt{2}}$ at $x=\frac{3\pi}{8}$ and $h(x)$ attains its maximum of $2\sqrt{2+\sqrt{2}}$ at $x=\frac{5\pi}{8}$. Hence

$$3\sin x+|\cos 3x|\le\sqrt{2+\sqrt{2}} \qquad (3)$$

for all $x\in[0,\pi]$ with equality if and only if $x=\frac{3\pi}{8}$ or $\frac{5\pi}{8}$.

Next, consider the function f defined on $\mathbb{R}^3$ by

$$f(\alpha,\beta,\gamma)=|\sin\alpha|+|\sin\beta|+|\sin\gamma|+|\cos(\alpha+\beta+\gamma)|.$$

Note that f is periodic with period π relative to each of the variables α,β,γ.

Hence, we could consider only those (α,β,γ) lying in the compact set $K=[0,\pi]^3$, so $f(\alpha,\beta,\gamma)=\sin\alpha+\sin\beta+\sin\gamma+|\cos(\alpha+\beta+\gamma)|$. Since f is continuous on a compact set, it attains its maximum at some point $(a,b,c)\in K$.

We claim that $a = b = c$. It clearly suffices to show that $a = b$. Suppose to the contrary that $a \neq b$. Then we may assume that $a < b$. Set $\theta = \frac{a+b}{2}$ and $\lambda = \frac{b-a}{2}$. Then $(\theta, \theta, c) \in K$, $a = \theta - \lambda, b = \theta + \lambda$ and $0 < \lambda < \pi$. Since $\cos\lambda < 1$ and $\sin\theta \neq 0$, we have

$$\begin{aligned} f(a,b,c) &= f(\theta-\lambda, \theta+\lambda, c) \\ &= \sin(\theta-\lambda) + \sin(\theta+\lambda) + \sin c + |\cos((\theta-\lambda)+(\theta+\lambda)+c)| \\ &= 2\sin\frac{(\theta-\lambda)+(\theta+\lambda)}{2}\cos\frac{(\theta-\lambda)-(\theta+\lambda)}{2} + \sin c + |\cos(2\theta+c)| \\ &= 2\sin\theta\cos\lambda + \sin c + |\cos(2\theta+c)| \\ &< 2\sin\theta + \sin c + |\cos(2\theta+c)| = f(\theta,\theta,c) \end{aligned}$$

contradicting the assumption that $f(a,b,c)$ is a maximum on K.

Hence, $f(\alpha,\beta,\gamma)$ attains its maximum over K when $\alpha = \beta = \gamma$ and this value is the maximum of $3\sin x + |\cos 3x|$ for $x \in [0,\pi]$ which is $2\sqrt{2+\sqrt{2}}$ by (3) and our proof is complete.

Math Quotes

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

Halmos, Paul R. in "I Want to be a Mathematician".