OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2013 : 39(9), p. 397–398.

OC141. Find all non-zero polynomials $P(x), Q(x)$ of minimal degree with real coefficients such that for all $x \in \mathbb{R}$ we have :

$$P(x^2) + Q(x) = P(x) + x^5 Q(x)$$

Originally from the Greece National Olympiad 2012 Problem 2.

We received three correct submissions. We present the solution by Titu Zvonaru and Neculai Stanciu.

Isolating for P and Q shows that

$$2\deg(P) = \deg(Q) + 5$$

which shows that the smallest possible degree for P is 3.

If $\deg(P) = 3$, then $\deg(Q) = 1$. Setting $P(x) = ax^3 + bx^2 + cx + d$ and $Q(x) = mx + n$ in the equation yields

$$ax^6 + bx^4 + cx^2 + d + mx + n = ax^3 + bx^2 + cx + d + mx^6 + nx^5$$

and when comparing the coefficient of x^3 yields that $a = 0$, contradicting the fact that $\deg(P) = 3$.

If $\deg(P) = 4$, then $\deg(Q) = 3$. Setting $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ and $Q(x) = mx^3 + nx^2 + px + q$ in the equation yields

$$\begin{aligned} &ax^8 + bx^6 + cx^4 + dx^2 + e + mx^3 + nx^2 + px + q = \\ &ax^4 + bx^3 + cx^2 + dx + e + mx^8 + nx^7 + px^6 + qx^5. \end{aligned}$$

Equating coefficients yields that $m = a$, $n = 0$, $b = p$, $q = 0$, $c = a$, $b = m$, $d = c$ and $d = p$. Hence, we have that

$$P(x) = ax^4 + ax^3 + ax^2 + ax + e \quad \text{and} \quad Q(x) = ax^3 + ax.$$

OC142. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y)\,;\, \forall x, y \in \mathbb{R}\,.$$

Originally from the Japan Mathematical Olympiad Problem 2.

We received four correct submissions. We present the solution by Joseph Ling.

It is easy to verify that $f(x) = x$ for all x is a solution to

$$f(f(x+y)f(x-y)) = x^2 - yf(y). \qquad (*)$$

We claim that it is the only solution.

Letting $x = y = 0$, we see that the number $z = f(0)^2$ satisfies $f(z) = 0$. Also,

$$f(0) = f(f(z+0)f(z-0)) = z^2 - 0f(0) = z^2.$$

Now, given any $y \in \mathbb{R}$, we let $x = y + z$. Then $f(x-y) = f(z) = 0$ and the given equation becomes

$$f(0) = (y+z)^2 - yf(y).$$

So,

$$yf(y) = (y+z)^2 - f(0) = (y+z)^2 - z^2 = y(y+2z).$$

It follows that for all $y \neq 0$, $f(y) = y + 2z$. In particular, if $z \neq 0$, then

$$0 = f(z) = z + 2z = 3z \Longrightarrow z = 0,$$

a contradiction. Therefore, $z = 0$. Consequently, $f(y) = y + 0 = y$ for all $y \neq 0$. But we also have $f(0) = z^2 = 0^2 = 0$. This completes the proof.

OC143. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

Originally from the Asian Pacific Mathematical Olympiad 2012 Problem 3.

We present the solution by Oliver Geupel.

For every prime p, the pair (p, p) is a solution. Moreover, $(2, 4)$ is a solution. We prove that there are no other solutions.

Note that the function $f(x) = \dfrac{\log x}{x}$ is decreasing for $x \geq e$.

The cases $p = 2$ with $n \leq 4$ are easily inspected. For $n \geq 5$ we deduce

$$\frac{\log 2}{2} = \frac{\log 4}{4} > \frac{\log n}{n};$$

whence $n \log 2 > 2 \log n$, so that $0 < \dfrac{n^2+1}{2^n+1} < 1$.

Suppose that (p, n) is a solution with $p \geq 3$. For $n > p$, we have

$$\frac{\log p}{p} > \frac{\log n}{n};$$

whence $0 < \dfrac{n^p+1}{p^n+1} < 1$, a contradiction. Thus

$$1 \leq n \leq p. \qquad (1)$$

Since the integer p^n+1 is even, the number n^p+1 is also even ; whence n is odd. As a consequence, we have the identity $p^n+1 = (p+1)(p^{n-1}-p^{n-2}+p^{n-3}-\cdots+1)$. Therefore $p+1$ is a divisor of n^p+1. Similarly, $p+1$ is a divisor of p^p+1. We obtain

$$n^p \equiv -1 \equiv p^p \pmod{p+1}\,. \tag{2}$$

It follows that the numbers n and $p+1$ are relatively prime. By Euler's Theorem, we obtain $n^{\varphi(p+1)} \equiv 1 \pmod{p+1}$. Applying the same theorem, we also get $p^{\varphi(p+1)} \equiv 1 \pmod{p+1}$. Consequently

$$n^{\varphi(p+1)} \equiv p^{\varphi(p+1)} \pmod{p+1}\,. \tag{3}$$

Lemma 1 *Let a, b, and m be integers such that $\gcd(a,m) = \gcd(b,m) = 1$ and suppose that k and ℓ are positive integers such that $a^k \equiv b^k \pmod{m}$ and $a^\ell \equiv b^\ell \pmod{m}$. Then it holds $a^{\gcd(k,\ell)} \equiv b^{\gcd(k,\ell)} \pmod{m}$.*

The numbers a and b are members of the abelian multiplicative group of congruence classes modulo m which are coprime to m. If, say, $k < \ell$, we obtain $a^{k-\ell} \equiv b^{k-\ell} \pmod{m}$. By the Euclidean algorithm, we arrive at the result after a finite number of repetitions of this argument. □

From (2) and (3) we deduce by the lemma that

$$n^{\gcd(p,\ \varphi(p+1))} \equiv p^{\gcd(p,\ \varphi(p+1))} \pmod{p+1}\,.$$

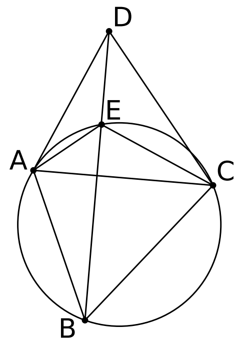
Clearly, $\varphi(p+1) < p$, so that $\gcd(p, \varphi(p+1)) = 1$ and $n \equiv p \pmod{p+1}$. In view of (1), we conclude $n = p$. The proof is complete.

OC144. Let $ABCD$ be a convex circumscribed quadrilateral such that $\angle ABC + \angle ADC < 180^\circ$ and $\angle ABD + \angle ACB = \angle ACD + \angle ADB$. Prove that one of the diagonals of quadrilateral $ABCD$ passes through the midpoint of the other diagonal.

Originally from Romania TST 2012 Day 2 Problem 2.

We present the solution by Oliver Geupel.

We prove the stronger statement that the quadrilateral $ABCD$ is a kite.



Because $\angle ABC + \angle ADC < 180^\circ$, the circle (ABC) meets the diagonal BD at an interior point E. By the inscribed angles theorem and by hypothesis, we have

$$\begin{aligned}\angle EAD = 180^\circ - \angle DEA - \angle ADE = \angle AEB - \angle ADE = \angle ACB - \angle ADB \\ = \angle ACD - \angle ABD = \angle ACD - \angle ECA = \angle DCE.\end{aligned}$$

Using the law of sines in triangles AED, CDE, and ABC, we get

$$\frac{AD}{\sin \angle DEA} = \frac{DE}{\sin \angle EAD} = \frac{DE}{\sin \angle DCE} = \frac{CD}{\sin \angle CED}$$

and

$$\begin{aligned}\frac{AB}{\sin \angle DEA} = \frac{AB}{\sin \angle AEB} = \frac{AB}{\sin \angle ACB} \\ = \frac{BC}{\sin \angle BAC} = \frac{BC}{\sin \angle BEC} = \frac{BC}{\sin \angle CED}.\end{aligned}$$

Hence,

$$AB \cdot CD = BC \cdot AD. \tag{1}$$

Since the quadrilateral $ABCD$ is circumscribed, we have

$$AB + CD = BC + AD. \tag{2}$$

From (1) and (2), we deduce that it holds either $AB = BC$ and $CD = AD$ or $AB = AD$ and $BC = CD$. Thus the quadrilateral $ABCD$ is a kite.

OC145. Let $n \geq 2$ be a positive integer. Consider an $n \times n$ grid with all entries 1. Define an operation on a square to be changing the signs of all squares adjacent to it but not the sign of its own. Find all n for which it is possible to find a finite sequence of operations which changes all entries to -1.

Originally from China Western Mathematical Olympiad 2012, Day 2 Problem 3.

There were no solutions submitted.