FOCUS ON...

No. 14

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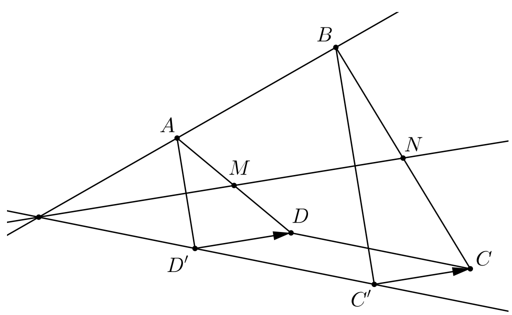
Solutions to Exercises from Focus On... No. 6 – 11

From Focus On... No. 6

(a) *Let A, B, C, D be four points in the plane such that $AB = CD$ and M, N be the midpoints of AD, BC, respectively. Show that the angle MN makes with the line AB equals the angle it makes with the line CD.*

Since $AB = CD$, there exists a unique glide reflection g such that $g(A) = D$ and $g(B) = C$ (g may reduce to a reflection). The axis of g is the line MN (since the axis passes through the midpoint of any segment joining a point to its image). It follows that $g = \mathbf{r} \circ \mathbf{t} = \mathbf{t} \circ \mathbf{r}$ where $\mathbf{r}$ denotes the reflection in MN and $\mathbf{t}$ is a translation whose vector $\overrightarrow{u}$, if not $\overrightarrow{0}$, is parallel to MN.



Let $C' = \mathbf{t}^{-1}(C)$ and $D' = \mathbf{t}^{-1}(D)$. We have

$$\mathbf{r}(C') = \mathbf{r} \circ \mathbf{t}^{-1}(C) = (\mathbf{t} \circ \mathbf{r})^{-1}(C) = g^{-1}(C) = B$$

and similarly, $\mathbf{r}(D') = A$. Thus, the line MN is an axis of symmetry of the lines AB and $C'D'$ and, as such, makes the same angle with each of them. The result follows since CD is parallel to $C'D'$.

(b) *If ABC is a triangle, find the axis and the vector of the glide reflection $\mathbf{r}_{AC} \circ \mathbf{r}_{BC} \circ \mathbf{r}_{AB}$ where $\mathbf{r}_{XY}$ denotes the reflection in the line XY.*

The reader is referred to problem **3789**, solution 1 [2013 : 427].

From Focus On... No. 7

(a) *Consider the sums $S_n(m) = \sum_{i=1}^{n} \frac{w_i^m}{D'(w_i)}$ where $D(x) = \prod_{i=1}^{n}(x - w_i)$ and suppose $w_i \neq 0$ for $i = 1, 2, \ldots, n$. Calculate $S_n(-1)$ and $S_n(-2)$.*

Recall the equality $\frac{1}{D(x)} = \sum_{i=1}^{n} \frac{1}{D'(w_i)} \cdot \frac{1}{x-w_i}$. We readily deduce

$$S_n(-1) = \sum_{i=1}^{n} \frac{1}{w_i D'(w_i)} = -\frac{1}{D(0)} = \frac{(-1)^{n+1}}{w_1 \cdot w_2 \cdot \cdots \cdot w_n}.$$

Now, differentiating both sides of the equality, we obtain

$$\frac{D'(x)}{(D(x))^2} = \sum_{i=1}^{n} \frac{1}{(x-w_i)^2 D'(w_i)}$$

so that

$$S_n(-2) = \sum_{i=1}^{n} \frac{1}{w_i^2 D'(w_i)} = \frac{D'(0)}{(D(0))^2}.$$

Since $(D(0))^2 = (w_1 \cdot w_2 \cdot \cdots \cdot w_n)^2$ and $D'(0) = (-1)^{n-1} \sum_{i=1}^{n} \left(\prod_{k=1, k \neq i}^{n} w_k \right)$, we finally get

$$S_n(-2) = \frac{(-1)^{n-1}}{w_1 \cdot w_2 \cdot \cdots \cdot w_n} \cdot \sum_{i=1}^{n} \frac{1}{w_i}.$$

(b) *Using the decomposition of* $\frac{1}{x^n-1}$, *rework problem* ***2657*** [2001 : 336 ; 2002 : 401], *that is prove that*

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k} \right) = \frac{2k}{1+(-1)^{k+1}\sqrt{2}\sin\theta}.$$

We recall the decomposition

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\omega^j x - 1}, \tag{1}$$

where $\omega = \exp(-2\pi i/n)$. We shall also make use of the following formula

$$2i \left(\frac{1}{e^{i\alpha}+1} - \frac{1}{e^{i\beta}+1} \right) = \tan\frac{\alpha}{2} - \tan\frac{\beta}{2}, \tag{2}$$

which is easily verified (note that $\tan t = -i \cdot \frac{e^{2it}-1}{e^{2it}+1} = \frac{2i}{e^{2it}+1} - i$).

Returning to the problem, we set $z_1 = -\exp\left(\frac{i(\theta - 3\pi/4)}{k}\right)$, $z_2 = -\exp\left(\frac{i(\theta-\pi/4)}{k}\right)$ and first suppose that k is even. Since

$$\frac{1}{1-\sqrt{2}\sin\theta} = \frac{i}{e^{i(\theta-3\pi/4)}-1} - \frac{i}{e^{i(\theta-\pi/4)}-1},$$

(1) yields

$$\frac{2k}{1-\sqrt{2}\sin\theta} = 2ki\left(\frac{1}{z_1^k - 1} - \frac{1}{z_2^k - 1} \right) = 2i \sum_{j=0}^{k-1} \left(\frac{1}{\omega^j z_1 - 1} - \frac{1}{\omega^j z_2 - 1} \right).$$

(Here ω denotes $\exp(-2\pi i/k)$.) But, with the help of (2), we obtain

$$\begin{aligned} 2i\left(\frac{1}{\omega^j z_1 - 1} - \frac{1}{\omega^j z_2 - 1}\right) \\ &= \tan\left(\frac{\theta - \pi/4 - 2\pi j}{2k}\right) + \tan\left(\frac{3\pi/4 - \theta + 2\pi j}{2k}\right) \\ &= \tan\left(\frac{4\theta + \pi(4(2(k-j)) - 1)}{8k}\right) + \tan\left(\frac{\pi(4(2j+1) - 1) - 4\theta}{8k}\right) \end{aligned}$$

and so

$$\frac{2k}{1 - \sqrt{2}\sin\theta} = \sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right).$$

The calculation is similar when k is odd. We have $\frac{2k}{1+\sqrt{2}\sin\theta} = 2ki\left(\frac{1}{\overline{z_2}^k - 1} - \frac{1}{\overline{z_1}^k - 1}\right)$. As above, we deduce that

$$\frac{2k}{1 + \sqrt{2}\sin\theta} = \sum_{j=0}^{k-1}\left(\tan\left(\frac{\pi(4(2(k-j)+1) - 1) - 4\theta}{8k}\right) + \tan\left(\frac{4\theta + \pi(4(2j) - 1)}{8k}\right)\right)$$

and the result follows.

(c) *Problem **3140*** [2006 : 238, 240 ; 2007 : 243] *required a proof of the inequality* $\prod_{k=1}^{n} a_k^{\frac{1}{p_k}} < 1$ *where* $n \geq 2, a_1, \ldots, a_n > 0$ *and* $p_k = \prod_{j \neq k}(a_j - a_k)$. *Find an alternative to Walther Janous's featured proof.*

We mimic the method developed in the column and omit the details.

Let

$$A(x) = \frac{1}{(x+a_1)(x+a_2)\cdots(x+a_n)}$$

whose decomposition into partial fractions is

$$A(x) = \sum_{i=1}^{n} \frac{1}{p_i} \cdot \frac{1}{x + a_i}.$$

Using $\sum_{i=1}^{n} \frac{1}{p_i} = pp(xA(x)) = 0$, we easily obtain

$$\int_0^\infty A(x)\,dx = -\sum_{i=1}^{n} \frac{1}{p_i}\,\ln(a_i).$$

Since $\int_0^\infty A(x)\,dx > 0$, we see that $\sum_{i=1}^{n} \frac{1}{p_i}\,\ln(a_i)$ must be negative and the desired inequality follows.

From Focus On... No. 8

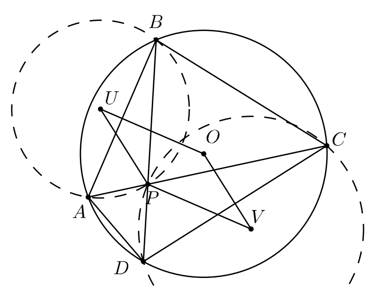
1. *Two circles, Γ with diameter AB, and Δ with centre A, intersect at points C and D. The point M (distinct from C and D) lies on Δ. The lines BM, CM and DM intersect Γ again at N, P and Q, respectively. Show that MN is the geometric mean of NC and ND.*

This is question 2 of **2666**. We keep the notations and figure of question 1 solved in Focus On... No 8. In particular, $\mathbf{I}$ denotes the inversion with centre M exchanging A and R. Since $N = \mathbf{I}(B)$ and $C = \mathbf{I}(P)$, we have

$$NC = \frac{|p|BP}{MB \cdot MP} = \frac{MB \cdot MN \cdot BP}{MB \cdot MP} = MN \cdot \frac{BP}{MP}.$$

In a similar way, $ND = MN \cdot \frac{BQ}{MQ}$. Now, because $MPBQ$ is a parallelogram, we have $BP = MQ$ and $BQ = MP$. It follows that $\frac{NC}{MN} = \frac{MN}{ND}$ and therefore $MN = \sqrt{NC \cdot ND}$.

2. *Let A, B, C, and D be points on a circle with centre O. If AB is not parallel to CD and U, V are the circumcentres of $\Delta APB, \Delta CPD$, prove that $OUPV$ is a parallelogram.*



Let $\mathbf{I}$ denote the inversion with centre P whose power is the power of P with respect to the circle Γ passing through A, B, C, D. Since $\mathbf{I}(A) = C$ and $\mathbf{I}(B) = D$, $\mathbf{I}$ transforms the circle (APB) into the line CD. It follows that PU is perpendicular to CD and so is parallel to the perpendicular bisector OV of CD. Similarly, PV is parallel to OU. Thus, $OUPV$ is a parallelogram (note that O, U, P, V are not collinear since otherwise AB and CD would be parallel).

From Focus On... No. 10

The following limits were to be evaluated in **3604** and in **3642** :

$$\lim_{n\to\infty} \frac{\int_0^1 (x^2 - x - 2)^n \, dx}{\int_0^1 (4x^2 - 2x - 2)^n \, dx} \quad \text{and} \quad \lim_{n\to\infty} \frac{\int_0^1 (2x^2 - 5x - 1)^n \, dx}{\int_0^1 (x^2 - 4x - 1)^n \, dx} \ .$$

It is easily checked that each of the functions $x \mapsto -x^2+x+2$ and $x \mapsto -4x^2+2x+2$ is positive and attains its maximum on $[0,1]$. From the case (c) of the last paragraph of the column, we deduce

$$\int_0^1 (-x^2+x+2)^n\,dx \;\sim\; \sqrt{\frac{\pi}{n}}\left(2+\frac{1}{4}\right)^{n+\frac{1}{2}} \quad \text{and}$$
$$\int_0^1 (-4x^2+2x+2)^n\,dx \;\sim\; \sqrt{\frac{\pi}{4n}}\left(2+\frac{4}{16}\right)^{n+\frac{1}{2}}$$

as $n \to \infty$. It readily follows that the first required limit is 2.

Each of the functions $x \mapsto -2x^2+5x+1$ and $x \mapsto -x^2+4x+1$ is positive and strictly increasing on $[0,1]$. From the case (a) this time, we obtain

$$\int_0^1 (-2x^2+5x+1)^n\,dx \;\sim\; \frac{(-2+5+1)^{n+1}}{n(2\cdot(-2)+5)} \quad \text{and}$$
$$\int_0^1 (-x^2+4x+1)^n\,dx \;\sim\; \frac{(1+4-1)^{n+1}}{n(2\cdot(-1)+4)}$$

as $n \to \infty$. Again, the desired limit is 2.

From Focus On... No. 11

1. *Find* ρ, α *and* $\ell > 0$ *such that* $\displaystyle\lim_{n\to\infty} \rho^n n^\alpha \sum_{k=1}^n \frac{5^n}{n\binom{2n-1}{n}} = \ell$.

Let $a_n = \frac{5^n}{n\binom{2n-1}{n}}$. A short calculation gives $\frac{a_{n+1}}{a_n} = \frac{5n}{2(2n+1)}$ and it follows that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{5}{4} > 1$. From the first of the three results of the column,

$$\sum_{k=1}^n a_k \;\sim\; \frac{5/4}{5/4-1}\cdot a_n = 5a_n \quad \text{as} \quad n \to \infty.$$

With the help of Stirling's formula $n! \;\sim\; n^n e^{-n}\sqrt{2\pi n}$, we find

$$n\binom{2n-1}{n} = \frac{n}{2}\cdot\frac{(2n)!}{(n!)^2} \;\sim\; \frac{4^n\cdot\sqrt{n}}{2\sqrt{\pi}}$$

so that $a_n \;\sim\; \left(\frac{5}{4}\right)^n \cdot \frac{2\sqrt{\pi}}{\sqrt{n}}$ and

$$\rho^n n^\alpha \sum_{k=1}^n a_k \;\sim\; 10\sqrt{\pi}\left(\frac{5\rho}{4}\right)^n \cdot n^{\alpha-\frac{1}{2}}$$

as $n \to \infty$.

We can now conclude : $\rho^n n^\alpha \sum\limits_{k=1}^n a_k$ has a finite nonzero limit as $n \to \infty$ if and only if $\rho = \frac{4}{5}$ and $\alpha = \frac{1}{2}$, in which case the limit is $\ell = 10\sqrt{\pi}$.

2. *Find α for the following sequence to be convergent*

$$\left(\frac{\sum_{k=1}^{n+1} k!\csc(\pi/2^k)}{\sum_{k=1}^{n} k!\csc(\pi/2^k)} - n\alpha\right)_{n\geq 1}.$$

What is its limit in that case ?

Let $a_n = n!\csc(\pi/2^n)$. We easily obtain

$$\frac{a_{n+1}}{a_n} = 2(n+1)\cos(\pi/2^{n+1})$$

and deduce that $\frac{a_{n+1}}{a_n} \sim 2n$ as $n \to \infty$. Furthermore,

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n} - 2n\right) = \lim_{n\to\infty}\left(2\cos(\pi/2^{n+1}) - 2n\left(1-\cos(\pi/2^{n+1})\right)\right) = 2.$$

Note that $\lim_{n\to\infty}\left(2n\left(1-\cos(\pi/2^{n+1})\right)\right) = 0$ since $1-\cos(\pi/2^{n+1}) \sim \frac{\pi^2}{2^{2n+3}}$ as $n \to \infty$.

From the third result proved in the column, we see that the given sequence is convergent when $\alpha = 2$ and its limit then is 2.

A lot of information out of nothing

Mathematician R said the following to mathematicians P and S : "I thought of two natural numbers. They are each greater than 1 and their sum is less than 100. I will secretly tell mathematician P their product and I will secretly tell mathematician S their sum." He did just that and asked mathematicians P and S to guess the numbers. The following dialogue took place :

P : I cannot tell what the numbers are.
S : I knew you couldn't.
P : Then I know what they are.
S : Then so do I.

Can you guess the numbers ?

Originally from article "Many bits out of nothing" by S. Artemov, Y. Gimatov and V. Fedorov, Kvant 1977 (3).