From the Adequality of Fermat to Nonstandard Analysis

Paul Deguire

Pierre de Fermat was a 17th century French mathematician. Although he worked full time as a lawyer at the *parlement* (provincial court) of Toulouse, he found enough time for high-quality mathematical research in many areas. One of his interests was a sort of early differential calculus. His main technique was a concept of approximation that he called *"adégalité"*, which would only be made rigorous centuries later.

He adapted this word from the Latin *"adaequatio"* ("conformity, making equal") which he appears to have taken from Bachet de Méziriac's Latin translation of Diophantus' *Arithmetica* — presumably, the famous copy with margins too small to contain the supposed proof of his "Last Theorem". Diophantus, who wrote in Greek around 250 AD, actually used the word $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$, by which he meant an exact process for finding solutions to number theory problems, known to lie close to a certain value (see [H], pages 95-97, 206-207). Except that it involved small numbers, it had little in common with Fermat's technique.

Fermat used this notion, which we will translate by "adequality", to find things such as extrema or the slopes of tangent lines. This was in the 1630s, three decades before Christian Huygens, presenting the work of Fermat at the *Académie Francaise*, created the expression "infinitely small" (1667), and five decades before Newton and Leibniz wrote about calculus. The terms "function" (Euler) and "derivative" (Lagrange) were not introduced until the 18th century : Fermat referred to "curves" and "slopes of tangent lines," and his optimization problems were not expressed in terms of functions, as we would do today.

Infinitely small quantities had been used before Fermat to calculate areas and volumes. Archimedes, for instance, used them to calculate the volume of the sphere. After the independent invention of analytic geometry by Descartes and Fermat in the early 17th century, the study of curves became an important branch of mathematics, and determining their extrema and their tangent lines became important research topics. Fermat was the first to solve such problems using infinitesimals, and until Weierstrass formulated a rigorous definition of limit in the 19th century, this was the standard technique.

Here's an example of one of Fermat's results using adequality, from a text published in 1636 [T]. Given a segment AB, where should we put the point C such that the rectangle on AC and CB has a maximal area ?

Let the length of AB be a and the length of CB be x ; we want to maximize $x(a-x)$. Suppose that C has been properly located and that our maximum is $b(a-b)$. Fermat knew that Oresme (14th century) and Kepler (early 17th century) had both observed that near a maximum the value of such expressions change very little. Today we would say that, near a maximum, the derivative is very small :

Fermat expressed this by saying that the values were adequal.

Fermat supposed that e is a very small quantity so that $(b+e)(a-(b+e))$ would then be very close to $b(a-b)$. He wrote : $(b+e)(a-(b+e)) \sim b(a-b)$, the symbol $\sim$ meaning that the quantities are adequal. Therefore,

$$ab - b^2 - be + ae - be - e^2 \sim ab - b^2.$$

Eliminating common terms, Fermat was left with

$$ae \sim 2be + e^2.$$

He then divided by e, obtaining

$$a \sim 2b + e$$

and let $e = 0$, obtaining $a = 2b$. Thus the point C is the midpoint of AB, and the rectangle with given perimeter and largest area is a square.

Similar use of infinitely small quantities worked well during the first 200 years of differential calculus, not only to find extrema but to calculate derivatives and their applications.

But even an average modern student would ask : how can you divide by e if e is equal to 0 ? The legitimate objection was made by other mathematicians and by philosophers. But the methods obviously worked, providing fantastic new results not obtainable otherwise. Better yet, as Newton showed, the physical world agreed with the new calculus as well. So the method was adopted, despite its lack of rigour.

Today we would say that Fermat was *essentially* solving

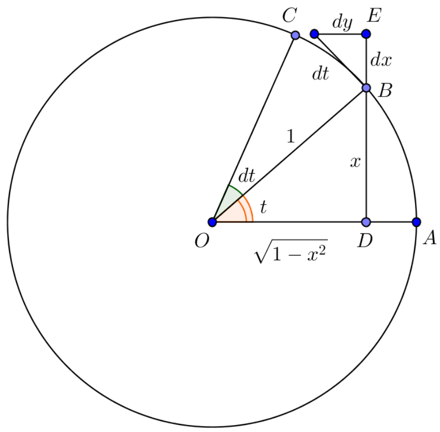
$$\lim_{e\to 0} \frac{f(b+e) - f(b)}{e} = 0$$

where the function f gives the area of the rectangle : that is, setting the derivative equal to 0. We know what a limit is, and we can provide rigorous analytical arguments. But our actual calculation exactly parallels what Fermat did four centuries ago ; and the student who only knows how to grind out the calculation knows no more than Fermat knew at the time.

The next two examples, from Gottfried Wilhelm Leibniz and Guillaume François Antoine, marquis de l'Hospital, give an idea of how the notion of "infinitely small number" was used in the first years of the new calculus.

Leibniz and the differential triangle

In the following diagram the circle is of radius 1, the angle is t, dt is a small increment of t and $x = \sin t$. Leibniz works with the small triangle $\triangle BEC$ whose sides are differentials and therefore infinitely small quantities. Therefore, he can consider the side BC, tangent to the circle at B, to be equal to the little arc dt. We know that this is only a good approximation, but the smaller dt is, the better is the approximation.



All quotients of pairs of sides in this triangle are indeterminate of the form $0/0$. The opponents of the new methods said that these quotients had no meaning. Leibniz had no problem because this infinitely small triangle $\triangle BEC$ is similar to the finite triangle $\triangle ODB$, therefore each quotient of sides of the infinitesimal triangle is equal to a well-defined quotient of sides of a finite triangle. For example, the indeterminate quotient dx/dt is equal to the well-defined finite quotient $\sqrt{1-x^2}/1$. Because $x = \sin t$, then $\sqrt{1-x^2} = \cos t$, and we obtain the now-familiar

$$\frac{d}{dt}\sin t = \cos t.$$

Similarly, $dy = \dfrac{xdx}{\sqrt{1-x^2}}$; using this last expression for dy and the Pythagorean Theorem, $dt^2 = dx^2 + dy^2$, Leibniz found the differential equation

$$\frac{d^2}{dt^2}x(t) = -x(t)$$

from which he was able to find the infinite McLaurin series for $x = \sin t$(see [Ka], pages 321-323). As we can see, he obtained this significant result using only the ordinary laws of geometry and infinitesimals.

Nobody was able then to define properly what an infinitely small quantity was, but the new calculus had so much success in both pure and applied mathematics that nothing stopped its evolution. For most of the next century it would be expanded and applied, although the basic notion of limit was at best a good intuition and the notion of infinitesimal had no solid ground to stand on.

The first differential calculus book

One of the first followers of Leibniz, and himself an important contributor to the new calculus, was Jean Bernoulli. The Marquis de l'Hospital, a French nobleman interested in mathematics, learned the new calculus from Bernoulli and wrote in 1696 the first calculus textbook, entitled *L'analyse des infiniments petits pour* *l'intelligence des lignes courbes* (see [L], page 23). What we now call “l'Hospital rule” – though it appears to have been Bernoulli's discovery – was published here for the first time.

In this text l'Hospital considers differences in variable quantities : what we today would call differentials. For most applications these differences have to be considered to be infinitely small. The rules of derivation are easy to follow (although we wouldn't speak about derivatives for another century). For instance to find the difference of a product xy, l'Hospital simply calculated

$$d(xy) = (x + dx)(y + dy) - xy.$$

Therefore, he had $d(xy) = xdy + ydx + dxdy$. Because $dxdy$ was negligible compared to the other two terms of the right side, he concluded that $d(xy) = xdy + ydx$.

Modern views of infinitely small quantities

In the 19th century, Augustin-Louis Cauchy would define properly the notions of limit and convergence and he had a definition for infinitely small as well that he frequently used. Following the developments of mathematics in the second half of the 19th century, notably the creation of mathematical logic, a mathematical definition of the limit was provided by Weierstrass. The use of quantifiers made it possible to say mathematically what Cauchy was saying in ordinary language. The mathematical definition could be worked with in full rigor, the mathematics were on solid ground again. The use of infinitely small quantities became useless and was somewhat forgotten.

Then, in 1960, Abraham Robinson wrote about a new notion, Nonstandard Analysis, based on an extension of the reals called “hyperreals”. This number system obeys the axioms of the real numbers, except the Archimedean axiom ; proving that it is consistent (or at least as consistent as the real number system) requires some subtle mathematical logic. It contains all real numbers, and also some “infinite” numbers so large that you cannot count to them ; and the reciprocals of those infinite numbers are infinitesimal. Infinitesimals lie above all negative reals and below all positive reals, but are not equal to 0. If you can count to (or past) a hyperreal number, then it is “finite” : any finite hyperreal is the sum of a standard real number and an infinitesimal.

Calculus can be done on the nonstandard reals, often quite easily as the notion of convergence is greatly simplified ; and Keisler [K] has even written an elementary calculus textbook using this approach.

However, it should be stressed that any result about the standard reals that can be obtained with nonstandard analysis can be obtained without it as well ; so most mathematicians continue to use classical standard analysis. Nevertheless, nonstandard analysis is sound, and demonstrates that the language of the infinitesimals, as used by Fermat, Leibniz and Cauchy, is fully compatible with mathematical rigour.

Perhaps the final chapter in this story is *synthetic differential geometry* and particularly *smooth infinitesimal analysis*, based on ideas developed by W. Lawvere in the 1960s. This approach to analysis uses infinitesimal elements that are nilpotent (they obey $e^2 = 0$ exactly), and typically avoids introducing infinities. It does, however, use logic without the excluded middle : for an infinitesimal e, the proposition $e = 0$ is neither true nor false! Unlike nonstandard analysis, smooth infinitesimal analysis has results that are not true classically. For instance, all functions of smooth infinitesimal analysis are differentiable! A good introduction to this is given by Bell [B].

References

[B] Bell, J. L. *An Invitation to Smooth Infinitesimal Analysis* (n.d.), online at `publish.uwo.ca/~jbell/invitationtoSIA.pdf`

[He] Heath, Sir Thomas L., *Diophantus of Alexandria : a study in the history of Greek algebra*, (Cambridge, 1910)

[Ka] Katz, Victor J., *A History of Mathematics*, brief edition, (Addison Wesley, 2004)

[Ke] Keisler, H. Jerome, *Elementary Calculus : An Infinitesimal Approach*, online at `https://www.math.wisc.edu/~keisler/calc.html`

[L] L'Hospital, Guillaume-Franois Antoine de, *Analyse des infiniment petits pour l'intelligence des lignes courbes*, (Paris, 1696), online at `http://gallica.bnf.fr/ark:/12148/bpt6k205444w`

[T] Tannery, Paul and Henry, Charles, Oeuvres de Fermat, online at `https://www.bibnum.education.fr/mathematiques/analyse/methode-pour-la-recherche-du-minimum-et-du-maximum`

The author thanks Robert Dawson for valuable comments and suggestions.

Paul Deguire
Université de Moncton
Moncton, NB