OLYMPIAD SOLUTIONS

OC136. $ABCD$ is a quadrilateral inscribed in a circle with centre $O$. If $AB = \sqrt{2} + \sqrt{2}$ and $\angle AOB = 135^\circ$, find the maximum possible area of $ABCD$.

Originally from the India National Olympiad 2012 Problem 1.

We received three correct submissions and one incorrect solution. We present the solution by Oliver Geupel.

We prove that the maximum possible area of quadrilateral $ABCD$ is

$$\frac{1}{8} \sqrt{2}(5 + 3\sqrt{3}). \quad (1)$$

Let $r$ be the radius of the circle $(O)$. Inspecting the triangle $ABO$, we find that

$$r^2 = \frac{AB^2}{4\cos^2 \angle BAO} = \frac{AB^2}{2(1 + \cos(180^\circ - \angle AOB))} = \frac{2 + \sqrt{2}}{2(1 + \cos 45^\circ)} = 1,$$

so that $r = 1$.

For the moment suppose that $O$ is an interior point of the quadrilateral $ABCD$.

Let $\alpha = \angle BOC, \beta = \angle COD, \gamma = \angle DOA$. Then, $\alpha, \beta, \gamma \in (0, \pi), \alpha + \beta + \gamma = 5\pi/4,$ and $2[ABCD] = \sin 135^\circ + \sin \alpha + \sin \beta + \sin \gamma$. Since the sine function is concave on the interval $[0, \pi]$, we obtain by Jensen’s inequality that

$$2[ABCD] \leq \sin 45^\circ + 3 \left( \frac{\alpha + \beta + \gamma}{3} \right)$$

$$= \sin 45^\circ + 3 \sin 75^\circ$$

$$= \sin 45^\circ + 3(\cos 45^\circ \sin 30^\circ + \sin 45^\circ \cos 30^\circ)$$

$$= \frac{1}{2} \sqrt{2} + 3 \left( \frac{1}{2} \sqrt{2} \cdot \frac{1}{2} + \frac{1}{2} \sqrt{2} \cdot \frac{1}{2} \right)$$

$$= \frac{1}{4} \sqrt{2}(5 + 3\sqrt{3}).$$

The equality holds if and only if $\alpha = \beta = \gamma = 5\pi/12$. Consequently, the maximum area of quadrilaterals $ABCD$ with interior point $O$ is as in (1).

Next suppose that $O$ is not an interior point of $ABCD$. Then,

$$[ABCD] < \frac{\pi}{2} < \frac{1}{8} \sqrt{2}(5 + 3\sqrt{3}).$$

This completes the proof.

OC137. We denote by $S(k)$ the sum of the digits in the decimal representation of $k$. Prove that there are infinitely many positive integers $n$ for which

$$S(2^n + n) < S(2^n).$$

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Originally from the Poland Math Olympiad 2012 Day 2 Problem 3.

There was one correct submission and one incorrect submission. We present the solution by Oliver Geupel.

For any integer \( m \geq 2 \) consider the number

\[
n = 10^m - 2 = \underbrace{9 \ldots 9}_m \text{ digits}.
\]

Let us write \( n \) in the form \( n = 4q + 2 \) with an integer \( q \). Then,

\[
2n = 2(4q + 2) = 8 + 4 \cdot 16q \equiv 4 \cdot 6 \equiv 4 \pmod{10}.
\]

Therefore, the sum of carry digits in the addition of the integers \( 2n \) and \( n \) according to the standard school method of adding column-by-column, right-to-left is at least \( m \). Note that every single carry unit causes the sum of digits of the result of the addition to decrease by 9. Hence, the decrease in the sum of digits is at least \( 9m \).

Thus,

\[
S(2^n + n) \leq S(2^n) + S(n) - 9m = S(2^n) + 9(m - 1) + 8 - 9m = S(2^n) - 1.
\]

Consequently, for every \( m \geq 2 \), the integer \( n \) has the required property.

**OC138.** Find all positive integers \( a, b, c, p \geq 1 \) such that \( p \) is a prime and

\[
a^p + b^p = p^c.
\]

Originally from the France TST 2012, Day 1, Problem 3.

We received 2 correct submissions. We present the solution by Oliver Geupel.

It is straightforward to verify that the following quadruples \((a, b, c, p)\) are solutions:

\((2^\alpha, 2^\alpha, 2^\alpha + 1, 2), (3^\alpha, 2 \cdot 3^\alpha, 3^\alpha + 2, 3),\) and \((2 \cdot 3^\alpha, 3^\alpha, 3^\alpha + 2, 3)\) where \( \alpha \) is a nonnegative integer. We prove that there are no other solutions.

Suppose that \((a, b, c, p)\) is a solution.

First consider the case \( p = 2 \). We have \( a = 2^\alpha a_1, b = 2^\beta b_1 \) with nonnegative integers \( \alpha, \beta \), and odd \( a_1, b_1 \). Then, \( 2^{2\alpha}a_1^2 + 2^{2\beta}b_1^2 = 2^c \); whence \( \alpha = \beta \) and \( a_1^2 + b_1^2 = 2^{c-2\alpha} \). Taking this modulo 4, we see that \( c - 2\alpha = 1, a_1 = b_1 = 1 \). This completes the case \( p = 2 \).

It remains to consider \( p \geq 3 \).

Let \( v_p(k) \) denote the exact exponent of the prime \( p \) in the factorization of the integer \( k \) into primes.

**Lemma 1 (Lifting The Exponent (LTE) Lemma)** Let \( p \) be an odd prime. For any two different integers \( x, y \) with \( p \nmid x \) and \( x \equiv y \pmod{p} \) and any positive integer \( n \), it holds \( v_p(x^n - y^n) = v_p(x - y) + v_p(n) \).

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Lemma 2  The sequence \((n^{1/(n-1)})_{n=3,4,5,...}\) is decreasing.

Proof. By the geometric - arithmetic mean inequality,

\[
\left(\frac{n+1}{n}\right)^{n-1} \cdot \frac{1}{n} < \left(\frac{(n-1) \cdot \frac{n+1}{n} + \frac{1}{n}}{n}\right)^n = 1.
\]

Hence, \((n+1)^{n-1} < n^n\) and \((n+1)^{1/n} < n^{1/(n-1)}\).

Returning to our problem, note that we have \(a = p^\alpha a_1\) and \(b = p^\beta b_1\) with non-negative integers \(\alpha, \beta\), and positive integers \(a_1\) and \(b_1\) which are not divisible by \(p\). Thus, \(\alpha = \beta\) and \(a_1^p + b_1^p = p^{\alpha-\rho}\). Hence, \(a_1 + b_1 \equiv a_1^p + b_1^p \equiv 0 \pmod{p}\). By LTE, \(v_p(a_1^p + b_1^p) = v_p(a_1 + b_1) + 1\). Therefore, \(a_1^p + b_1^p = (a_1 + b_1)p\). By the general means inequality, we have

\[
(a_1 + b_1) \cdot \left(\frac{a_1 + b_1}{2}\right)^{p-1} = 2 \cdot \left(\frac{a_1 + b_1}{2}\right)^p < a_1^p + b_1^p = (a_1 + b_1)p.
\]

Thus,

\[
\frac{a_1 + b_1}{2} < p^{1/(p-1)}.
\]

In the case \(p = 3\), we obtain \(a_1 + b_1 \leq 3\), i.e. \((a_1, b_1) \in \{(1, 2), (2, 1)\}\).

In the case \(p \geq 5\), Lemma 2 implies \(a_1 + b_1 < 2\sqrt{5}\). Consequently, \(a_1 + b_1 \leq 2\), \(a_1 = b_1 = 1\), i.e. \(a = b\), which is impossible.

OC139. The numbers 1, 2, ..., 50 are written on a blackboard. Each minute any two numbers are erased and their positive difference is written instead. At the end one number remains. Find all the values this number can take.


We received three correct submissions. We present the solution by Oliver Geupel.

We show that the possible values of the last number are 1, 3, 5, ..., 47, 49.

Consider the general problem with numbers 1, 2, ..., \(n\) initially on the blackboard and let \(L_n\) denote the range of the values of the last number. The total sum of the numbers on the blackboard is decreased by the even number \(a + b - |a - b| = 2\min\{a, b\}\) when erasing \(a\) and \(b\) and writing their positive difference. Hence each subsequent sum has the same parity as the initial sum \(1 + 2 + \cdots + n = n(n+1)/2\). We prove by mathematical induction on \(n \geq 1\) that

\[
L_n = \left\{k : 0 \leq k \leq n, \ k \equiv \frac{n(n+1)}{2} \pmod{2}\right\}.
\]

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The base cases $n = 1$ and $n = 2$ are straightforward. For the induction step, assume that $n \geq 3$ is such that for every $m < n$ it holds

$$L_m = \left\{ k : \ 0 \leq k \leq m, \ k \equiv \frac{m(m + 1)}{2} \pmod{2} \right\}.$$  

We are to prove (1) for this number $n$. Let $r$ denote the remainder of $n \pmod{4}$. Consider the values $r = 0, 1, 2, 3$ in succession.

Case $r = 0$. By induction, $L_{n-1} = \{0, 2, 4, \ldots, n-2\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$ because we can start reducing the set $\{1, 2, \ldots, n-1\}$ up to the one remaining number $k$ and finally process the numbers $k$ and $n$. Hence $2, 4, 6, \ldots, n \in L_n$. By induction, $1 \in L_{n-2}$, note that for every $k \in L_{n-2}$, we have $k - (n - (n - 1)) \in L_n$. Thus, $0 \in L_n$.

Case $r = 1$. By induction, $L_{n-1} = \{1, 3, 5, \ldots, n-2\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Hence, $2, 4, 6, \ldots, n - 1 \in L_n$. By induction, $1 \in L_{n-2}$. For every $k \in L_{n-2}$, we have $k - (n - (n - 1)) \in L_n$. Thus, $0 \in L_n$.

Case $r = 2$. By induction, $L_{n-1} = \{1, 3, 5, \ldots, n-1\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Therefore, $1, 3, 5, \ldots, n - 1 \in L_n$.

Case $r = 3$. By induction we have $L_{n-1} = \{0, 2, 4, \ldots, n - 1\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Hence, $1, 3, 5, \ldots, n \in L_n$.

This completes the proof by induction.

**OC140.** Let $ABC$ be an obtuse triangle with $\angle A > 90^\circ$. Let circle $O$ be the circumcircle of $ABC$. $D$ is a point on the segment $AB$ such that $AD = AC$. Let $AK$ be the diameter of circle $O$, and let $L$ be the point of intersection of $AK$ and $CD$. A circle passing through $D, K, L$ intersects the circle $O$ at $P \neq K$. Given that $AK = 2, \angle BCD = \angle BAP = 10^\circ$, prove that

$$DP = \sin\left(\frac{\angle A}{2}\right).$$

*Originally from the Korea Mathematical Olympiad Day 1 Problem 1.*

*We present the solution by Oliver Geupel.*

Let lines $AP$ and $BC$ intersect at point $E$. Let point $F$ be the second intersection of the internal bisector of $\angle A$ with $\Gamma$. Let $O$ be the centre of $\Gamma$.

We have

$$\angle C = \angle ACD + \angle DCB = \left(90^\circ - \frac{\angle A}{2}\right) + 10^\circ = 100^\circ - \frac{\angle A}{2},$$

$$\angle B = 180^\circ - \angle A - \angle C = 80^\circ - \frac{\angle A}{2}.$$
Because $DC \perp AF$ and $CE \perp FO$, it holds
\[
\angle KPF = \angle KAF = \angle AFO = \angle DCB = 10^\circ,
\]
\[
\angle DLK = \angle CLA = 90^\circ - \angle KAF = 80^\circ.
\]
Since the quadrilateral $DPKL$ is cyclic, we obtain
\[
\angle KPD = 180^\circ - \angle DLK = 100^\circ.
\]
Hence,
\[
\angle FPD = \angle KPD - \angle KPF = 100^\circ - 10^\circ = 90^\circ. \tag{1}
\]
We have
\[
\angle BCP = \angle BAP = 10^\circ,
\]
\[
\angle AFP = \angle ACP = \angle C + \angle BCP = \left(100^\circ - \frac{\angle A}{2}\right) + 10^\circ = 110^\circ - \frac{\angle A}{2},
\]
\[
\angle AFD = \angle CFA = \angle B = 80^\circ - \frac{\angle A}{2},
\]
\[
\angle DFP = \angle AFP - \angle AFD = 30^\circ. \tag{2}
\]
From (1) and (2), we deduce that
\[
DP = \frac{DF}{2} = \frac{CF}{2} = FO \cdot \sin \frac{\angle FOC}{2} = \frac{KA}{2} \cdot \sin \angle FAC = \sin \left(\frac{\angle A}{2}\right).
\]
This completes the proof.

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