CONTEST CORNER

SOLUTIONS

CC86. A hexagon, $H$, is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Each side of length 1 is between two sides of length 3 and, similarly, each side of length 3 is between two sides of length 1. Find the area of $H$.

*Originally 1998 W.J. Blundon Mathematics Contest, problem 10.*

*We received five correct submissions. We present the solution by John Heuver.*

Consider a hexagon $H = ABCDEF$ with $AB = CD = EF = 1$ and $BC = DE = FA = 3$. Let the diagonals $AD$ and $BE$ intersect at $X$. Since $BE \parallel AF$ and $AD \parallel FE$, it follows that the quadrilateral $AFEX$ is a parallelogram with $AX = FE = 1$ and $AF =XE = 3$. Similarly, $BCDX$ is a parallelogram with $CD = BX = 1$ and $BC = XD = 3$.

Thus, both triangles $ABX$ and $DEX$ are equilateral with sides of length 1 and 3, respectively. Since their altitudes are correspondingly $\frac{\sqrt{3}}{2}$ and $\frac{3\sqrt{3}}{2}$, the area of the trapezoid $ABEF = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}(4 + 3) = \frac{7}{4}\sqrt{3}$ and the area of the trapezoid $BCDE = \frac{1}{2} \cdot \frac{3\sqrt{3}}{2}(4 + 1) = \frac{15}{4}\sqrt{3}$.

This lets us conclude that the area of the hexagon $H$ is $\frac{11}{2}\sqrt{3}$.

CC87. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of $BE$ is $\theta$ and the angle $FEA$ is $\alpha$, where $F$ is the intersection of $AC$ and $BE$. Find $\theta$ and $\cos \alpha$.

*Originally 2004 W.J. Blundon Mathematics Contest, problem 10.*

*We received seven correct submissions. We present the solution by Matei Coiculescu, slightly modified by the editor.*
Since the pentagon $ABCDE$ is regular, the internal angles all equal $\frac{3 \cdot 180^\circ}{5} = 108^\circ$, and $1 = AB = EA$. Since $EA = AB$ the triangle $ABE$ is isosceles, which implies that

$$\alpha = \angle FEA = \angle BEA = \frac{1}{2}(180^\circ - 108^\circ) = 36^\circ.$$  

Similarly, triangle $ABC$ is isosceles, so that $\angle BAC = \angle BAF = 36^\circ = \alpha$. Thus $\angle EFA = 2\alpha = 72^\circ$ (since it is the external angle of $\triangle ABF$ at $F$). Since the triangles $FAB$ and $ABE$ are similar (having equal corresponding angles), we have $\frac{FB}{AB} = \frac{AE}{BE}$, or

$$FB = \frac{1}{\theta}.$$  

Observe that $\angle FAE = \angle BAE - \angle BAF = 108^\circ - 36^\circ = 72^\circ = \angle AFE$. Thus, $\triangle EAF$ is isosceles, so that $1 = EA = EF$. Consequently, since $EB = EF + FB$,

$$\theta = 1 + \frac{1}{\theta}.$$  

The positive solution of this equation is

$$\theta = \frac{1 + \sqrt{5}}{2}.$$  

Finally, the Law of Cosines applied to $\alpha$ in triangle $ABE$ gives

$$\cos \alpha = \frac{1 + \theta^2 - 1}{2\theta} = \frac{\theta}{2}.$$  

In summary,

$$\theta = \frac{1 + \sqrt{5}}{2} \text{ and } \cos \alpha = \frac{1 + \sqrt{5}}{4}.$$  

CC88. A cat is located at $C$, 60 metres directly west of a mouse located at $M$. The mouse is trying to escape by running at 7 m/s in a fixed direction. The cat, an expert in geometry, runs at 13 m/s in a suitable straight line path that will intercept the mouse as quickly as possible. Suppose that the mouse is intercepted after running a distance of $d_1$ metres in a particular direction. If the mouse had been intercepted after it had run a distance of $d_2$ metres in the opposite direction, show that $d_1 + d_2 \geq 14\sqrt{30}$.

Originally 2007 Canadian Open Mathematics Challenge, problem B4c).

We received two correct submissions. We present the solution by Titu Zvonaru and Neculai Stanciu.

Let $A$ be the point where the cat catches the mouse after the mouse has run the distance $d_1$, and let $B$ be the point where the cat catches the mouse after the mouse has run the distance $d_2$.
From $M$ to $A$, the mouse runs for $\frac{d_1}{7}$ seconds, and from $M$ to $B$, the mouse runs for $\frac{d_2}{7}$ seconds. It follows that the segment $CA$ has length

$$|CA| = \frac{13d_1}{7}$$

and the segment $CB$ has length

$$|CB| = \frac{13d_2}{7}.$$ 

By Stewart’s theorem, we have that

$$|CA|^2 \cdot |BM| - |CM|^2 \cdot |AB| + |CB|^2 \cdot |AM| = |AM| \cdot |BM| \cdot |AB|. \tag{3}$$

Then by (1) and (2), and since $|CM| = 60$, $|AM| = d_1$, $|BM| = d_2$, and $|AB| = d_1 + d_2$, (3) becomes

$$\frac{169d_1^2d_2}{49} - 3600(d_1 + d_2) + \frac{169d_1^2d_2^2}{49} = d_1 \cdot d_2 \cdot (d_1 + d_2)$$

and we have

$$120d_1d_2(d_1 + d_2) = 49 \cdot 3600(d_1 + d_2).$$

Therefore,

$$d_1d_2 = 49 \cdot 30. \tag{4}$$

Now by the inequality of arithmetic and geometric means,

$$d_1 + d_2 \geq 2\sqrt{d_1d_2}$$

where $\sqrt{d_1d_2} = \sqrt{49 \cdot 30}$ by (4), so that

$$d_1 + d_2 \geq 14\sqrt{30}$$

as required.

**CC89.** Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two-variable polynomial in $x$ and $y$, prove that it must be constant.

*Originally 2014 Sun Life Financial Repêchage Competition, problem 1.*

*No solutions to this problem were received.*

**CC90.** For a given $k > 0$, $n \geq 2k > 0$, consider the square $R$ in the plane consisting of all points $(x, y)$ with $0 \leq x, y \leq n$. Color each point in $R$ gray if $\frac{x}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of $n$ and $k$.

*Originally question 9 from the 2001 Stanford Math Tournament, Calculus.*

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We present the solution by Digby Smith.

Suppose $0 \leq y \leq k$. Then, $\frac{xy}{k} \leq \frac{xk}{k} = x \leq x + y$. That is, the portion of the square $R$ where $0 \leq y \leq k$ is coloured gray.

Similarly, suppose $0 \leq x \leq k$. Then, $\frac{xy}{k} \leq \frac{ky}{k} = y \leq x + y$. That is the portion of the square $R$ where $0 \leq x \leq k$ is coloured gray.

The next step is to determine the area of the blue region, $A_b$, contained in the portion of the square $R$ where $k < x$ and $y \leq n$. To start with, let

$$a = \frac{kn}{n-k} \text{ and } n = \frac{kx}{x-k}.$$  

Solving for $x$, we have

$$kx = n(x - k) \implies kn = x(n - k) \implies x = \frac{kn}{n-k},$$  

so that $x = a$. Similarly, let $\frac{ky}{y-k} = n$. Then solving for $y$, we have that $y = a$. The curve $y = \frac{kx}{x-k}$ intersects the line $y = n$ at the point $P = (a, n)$ and intersects the line $x = n$ at the point $Q = (n, a)$. We now make use of some basic properties. First, we have

$$(n - k)(n - a) = n(n - 2k),$$

since

$$(n - k)(n - a) = (n - k)\left(n - \left(\frac{kn}{n-k}\right)\right)$$
$$= (n - k)n \left(\frac{n - k - k}{n-k}\right)$$
$$= n(n - 2k).$$

Second, we have

$$\ln(n - k) - \ln(a - k) = 2 \ln\left(\frac{n-k}{k}\right),$$

since

$$\ln(n - k) - \ln(a - k) = \ln(n - k) - \ln\left(\frac{kn}{n - k} - k\right)$$
$$= \ln(n - k) - \ln\left(\frac{kn - k + k^2}{n-k}\right)$$
$$= \ln(n - k) - 2\ln(k) + \ln(n - k)$$
$$= 2 \ln\left(\frac{n-k}{k}\right).$$

Furthermore, the following basic inequality holds:

$$k < a \leq n.$$  

*Crux Mathematicorum*, Vol. 40(8), October 2014
Proof. Starting with $2k \leq n$, it follows that $2kn \leq n^2$ and $kn \leq n^2 - kn$, so that $kn \leq n(n-k)$. Since $n > k$, it follows that

$$\frac{kn}{n-k} \leq n$$

with $a \leq n$. Next, starting with $k^2 > 0$, we have $kn - k^2 < kn$ and $k(n-k) < kn$. Since $n > k$, it follows that

$$k < \frac{kn}{n-k}$$

with $k < a$. That is, $k < a \leq n$. 

Now suppose that $k < x$, $y \leq n$. Then if $x + y < \frac{xy}{x}$, we have

$$kx < xy - ky = (x-k)y$$

making

$$\frac{kx}{x-k} < y \quad \text{and also} \quad \frac{ky}{y-k} < x.$$

Applying (7), it follows that the points $P$ and $Q$ are contained in the portion of the square $R$ where $k < x$, $y \leq n$. Thus the portion of the square $R$ coloured blue is given by

$$\frac{kx}{x-k} \leq y \leq n$$

with $a \leq x \leq n$. If $a = n$ (when $n = 2k$), then $A_b = 0$ with the area of the gray region, $A_g$, being $A_g = n^2$. Otherwise, if $a \neq n$, then

$$A_b = \int_a^n \left( n - \frac{kx}{x-k} \right) dx$$

$$= \int_a^n n - k \left( 1 + \frac{k}{x-k} \right) dx$$

$$= \int_a^n (n-k) - k^2 \left( \frac{1}{x-k} \right) dx$$

$$= (n-k)(n-a) - k^2(\ln(n-k) - \ln(a-k)).$$

so that

$$A_b = n(n-2k) - 2k^2 \ln \left( \frac{n-k}{k} \right)$$

by (5) and (6). It follows that

$$A_g = n^2 - A_b = 2nk + 2k^2 \ln \left( \frac{n-k}{k} \right),$$

so that the area of the area of the gray region in terms of $n$ and $k$ is

$$2nk + 2k^2 \ln \left( \frac{n-k}{k} \right).$$