THE CONTEST CORNER
No. 27
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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by November 1, 2015, although late solutions will also be considered until a solution is published.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC131. Let $D$ be the point on the side $AC$ of triangle $ABC$. Circle of radius $\frac{2}{\sqrt{3}}$ is inscribed in triangle $ABD$ and touches $AB$ at a point $M$; circle of radius $\sqrt{3}$ is inscribed in triangle $BCD$ and touches $BC$ at a point $N$. Given that $|BM|=6$ and $|BN|=5$, find the lengths of sides of triangle $ABC$.

CC132. You are told the following four statements about natural numbers $n$ and $k$:

a) $n+1$ is divisible by $k$,
b) $n = 2k + 5$,
c) $n+k$ is divisible by 3,
d) $n+7k$ is a prime number.

Three of these statements are true and one is not. Find all possible pairs $(n,k)$.

CC133. Ten numbers, not necessarily unique, are written in a row. Then, under every number, we write how many numbers in this row are smaller than it. Can the second row be

a) $9002536366$?
b) $5611485801$?

CC134. Let two tangent lines from the point $M(1,1)$ to the graph of $y = \frac{k}{x}$, $k < 0$ touch the graph at the points $A$ and $B$. Suppose that the triangle $MAB$ is a right-angle triangle. Find its area and the value of constant $k$.

CC135. Consider the following two arithmetic progressions:

$log a, log b, log c$ and $log a - log 2b, log 2b - log 3c, log 3c - log a$.
Can the values $a, b, c$ be the lengths of the sides of a triangle? If so, find the interior angles of this triangle.

**CC131.** Soit $D$ un point sur le côté $AC$ du triangle $ABC$. On trace le segment $BD$. Le cercle inscrit dans le triangle $ABD$ a un rayon de $2/\sqrt{3}$ et il touche le segment $AB$ en $M$. Le cercle inscrit dans le triangle $BCD$ a un rayon de $\sqrt{3}$ et il touche le segment $BC$ en $N$. Sachant que $|BM| = 6$ et $|BN| = 5$, déterminer les longueurs des côtés du triangle $ABC$.

**CC132.** On nous dit que les entiers strictement positifs $n$ et $k$ satisfont aux quatre conditions suivantes:

a) $n + 1$ est divisible par $k$,

b) $n = 2k + 5$,

c) $n + k$ est divisible par 3,

d) $n + 7k$ est un nombre premier.

Sachant que les deux entiers satisfont à exactement trois des quatre conditions, déterminer tous les couples possibles $(n, k)$.

**CC133.** Dix nombres, pas nécessairement distincts, sont écrits dans une rangée. Au dessous de chaque nombre, on écrit combien des nombres de la première rangée sont inférieurs à ce nombre. Est-il possible que la deuxième rangée soit

a) 9 0 0 2 5 3 6 3 6 6?

b) 5 6 1 1 4 8 5 8 0 1?

**CC134.** Deux droites, issues du point $M(1, 1)$, sont tangentes à la courbe d’équation $y = k/x$ ($k < 0$) aux points $A$ et $B$. Sachant que le triangle $MAB$ est rectangle, déterminer la valeur de $k$ et l’aire du triangle.

**CC135.** On considère deux suites arithmétiques,

$\log a, \log b, \log c$ et $\log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a$.

Est-il possible que $a, b$ et $c$ soient les longueurs des côtés d’un triangle? Si oui, déterminer les mesures des angles intérieurs de ce triangle.
CONTEST CORNER

SOLUTIONS

CC81. Quadrilateral $ABCD$ has the following properties:

1. the mid-point $O$ of side $AB$ is the centre of a semicircle;
2. sides $AD$, $DC$ and $CB$ are tangent to this semicircle.

Prove that $AB^2 = 4AD \times BC$.

_Originally 1998 W.J. Blundon Mathematics Contest, problem 10._

We received six solutions in total; four solutions came with the assumption that the diameter of the semicircle is the segment $AB$, which was not stated in the problem.

We present the solution by Šefket Arslanagić.

Assume notation as on the diagram. Since $OA = OB$, $OE = OG$, and $\angle E = \angle G = 90^\circ$, we have $\triangle AOE \cong \triangle BOG$. Hence $\angle AOE = \angle BOG = x$. Because tangents to the circle from an external point are equal, we have $\triangle DOE \cong \triangle DOF$ and $\triangle COG \cong \triangle COF$. Hence $\angle DOE = \angle DOF = y$ and $\angle COF = \angle COG = z$.

Also $2x + 2y + 2z = 180^\circ$, so $x + y + z = 90^\circ$, therefore $\angle OAE = \angle OAD = y + z$, $\angle ADO = x + z$, $\angle OBG = \angle OBC = y + z$, and $\angle OCB = x + y$.

From here, we have $\triangle AOD \sim \triangle BCO$, which gives us

$$
\frac{AD}{OB} = \frac{OA}{BC} \iff OA \cdot OB = AD \cdot BC \iff \frac{1}{2} AB \cdot \frac{1}{2} AB = AD \cdot BC.
$$

Hence, $AB^2 = 4AD \cdot BC$.

CC82. For each positive integer $N$, an _Eden sequence_ from \{1, 2, 3, \ldots, N\} is defined to be a sequence that satisfies the following conditions:

1. each of its terms is an element of the set of consecutive integers \{1, 2, 3, \ldots, N\},
2. the sequence is increasing, and

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3. the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from \{1, 2, 3\} are

\[
\begin{array}{cccc}
1 & 3 & 1, 2 & 1, 2, 3
\end{array}
\]

For each positive integer \(N\), define \(e(N)\) to be the number of Eden sequences from \{1, 2, 3, \ldots, N\}. If \(e(17) = 4180\) and \(e(20) = 17710\), determine \(e(18)\) and \(e(19)\).

Originally 2012 Euclid Contest, problem 10b.

Solved by Titu Zeonaru and Neculai Stanciu. Below is a modified version of their solution.

We denote by \(ev(N)\) the number of even length Eden sequences and by \(od(N)\) the number of odd length Eden sequences for \(N\), so \(e(N) = ev(N) + od(N)\). We have \(ev(1) = 0\), \(od(1) = 1\) and \(ev(2) = od(2) = 1\). For larger \(N\) we obtain the following.

If \(N\) is even, we have:

- \(ev(N) = ev(N - 1) + od(N - 1)\), since an even length Eden sequence for \(N\) that doesn’t end on \(N\) is also an even length Eden sequence for \(N - 1\), while a sequence that ends on \(N\) becomes an odd length Eden sequence for \(N - 1\) when we remove \(N\);

- \(od(N) = od(N - 1)\), since an odd length Eden sequence for \(N\) cannot end on \(N\) and is therefore also an odd length Eden sequence for \(N - 1\).

If \(N\) is odd, we have:

- \(ev(N) = ev(N - 1)\);

- \(od(N) = od(N - 1) + ev(N - 1) + 1 = e(N - 1) + 1\), with the same argument as above, except that there is also the Eden sequence formed only by the number \(N\).

For even \(N\), we have

\[
e(N) = ev(N) + od(N) = ev(N - 1) + od(N - 1) = e(N - 1) + e(N - 2) + 1,
\]

which is the same as for odd \(N\):

\[
e(N) = od(N) + ev(N) = e(N - 1) + 1 + ev(N - 1) = e(N - 1) + e(N - 2) + 1.
\]

From this, we can calculate

\[
e(20) = e(19) + e(18) + 1 = 2e(18) + e(17) + 2,
\]

from which we obtain \(e(18) = 6764\) and \(e(19) = 10945\).

[We note that \(e(N) + 1 = F_{n+2}\) are the Fibonacci numbers, as can be shown using the recursion for \(e(N)\).]
CC83. A map shows all Beryls Llamaburgers restaurant locations in North America. On this map, a line segment is drawn from each restaurant to the restaurant that is closest to it. Every restaurant has a unique closest neighbour. (Note that if $A$ and $B$ are two of the restaurants, then $A$ may be the closest to $B$ without $B$ being closest to $A$.) Prove that no restaurant can be connected to more than five other restaurants.

*Originally problem B3b from the 2004 Canadian Open Mathematics Challenge.*

We present a slightly expanded solution by Titu Zvonaru and Neculai Stanciu.

Suppose $A$ is a restaurant that is connected to six restaurants $P_1, \ldots, P_6$. The points $A$, $P_1$, and $P_j$ cannot be collinear in this order, because $P_j$ would then not be connected to $A$. We assume that $P_1, \ldots, P_6$ is the order in which the points appear around $A$ in clockwise direction.

From the six angles of type $\angle P_i AP_{i+1}$ at least one must be less than or equal to $60^\circ$; assume $\angle P_1 AP_2$ is this angle. This yields that in the triangle $P_1 AP_2$ the side $P_1 P_2$ is not the greatest. Suppose that $AP_1$ is the greatest.

Then either $P_1 P_2 < AP_2 < AP_1$ or $AP_2 < P_1 P_2 < AP_1$. In both cases $P_1$ cannot be connected to $A$.

CC84. Let $m$ and $n$ be odd positive integers. Each square of an $m$ by $n$ board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of $m$ and $n$.

*Originally 2014 Canadian Mathematical Olympiad, problem 2.*

We present the solution by Titu Zvonaru and Neculai Stanciu.

Let $m = 2p - 1, n = 2q - 1$. We will start by assuming that $p, q \geq 2$.

A red-dominated row will have at least $q$ red squares and a blue-dominated column will have at least $p$ blue squares. Thus, in order to have $m$ red-dominated rows and at least $n - 1$ blue-dominated columns, there would need to be at least

$$(2p - 1)q + (2q - 2)p = 4pq - q - 2p$$

squares in the board. However, there are

$$(2p - 1)(2q - 1) = 4pq - 2p - 2q + 1$$

squares on the board. And since $p, q \geq 2$ we have

$$4pq - q - 2p > 4pq - 2p - 2q + 1.$$}

Thus, we cannot have $m + n - 1$ or more as the sum.

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To see that we can obtain a sum of \( m + n - 2 \), consider a board formed by tiling the upper left \((m - 1) \times (n - 1)\) subgrid in a red and blue checkerboard pattern and then colouring the last square in each row red and the last column in each row blue. The bottom right corner can be coloured either colour. Then the first \( m - 1 \) rows are red-dominated and the first \( n - 1 \) columns are blue-dominated.

When \( m = 1 \), we can have \( n \) blue-dominated columns by colouring all squares blue. If we have \( n \) blue-dominated columns, we cannot have a red-dominated row, so this is a maximum.

Thus, the maximum is \( \max(m,n) \) if \( m \) or \( n \) is 1, and \( m + n - 2 \) otherwise.

**CC85.** While Lino was simplifying the fraction \( \frac{A^3 + B^3}{A^3 + C^3} \), he cancelled the threes \( A^3 \) and \( C^3 \) to obtain the fraction \( \frac{A + B}{A + C} \). If \( B \neq C \), determine a necessary and sufficient condition on \( A, B \) and \( C \) for Lino’s method to actually yield the correct answer, i.e., for \( \frac{A^3 + B^3}{A^3 + C^3} = \frac{A + B}{A + C} \).

Originally 2005 University of Waterloo Big E contest, problem 1.

We have received several incomplete or incorrect submissions. We present the solution by Michel Bataille.

In order for the fractions to be defined, we need to have \( A \neq -\omega C \), where \( \omega \) is a cube root of unity.

The equation \( \frac{A^3 + B^3}{A^3 + C^3} = \frac{A + B}{A + C} \) is equivalent to

\[
(A^3 + B^3)(A + C) = (A^3 + C^3)(A + B),
\]
\[
(A + B)(A + C)((A^2 - AB + B^2) - (A^2 - AC + C^2)) = 0,
\]
\[
(A + B)(A + C)(B - C)(B + C - A) = 0.
\]

Since we are given \( B \neq C \) and we assumed that \( A \neq -\omega C \), we are left with \( A = -B \) or \( A = B + C \).

When \( A = -B \) we get both fractions are 0, and so they are equal.

If \( A = B + C \), we get

\[
\frac{A^3 + B^3}{A^3 + C^3} = \frac{(2B + C)(B^2 + BC + C^2)}{(2C + B)(B^2 + BC + C^2)}.
\]

Note that we cannot have \( B^2 + BC + C^2 = 0 \) since this would require \( B = C = 0 \) or \( A = -\omega C \), neither of which are permitted. It follows that

\[
\frac{A^3 + B^3}{A^3 + C^3} = \frac{2B + C}{2C + B} = \frac{A + B}{A + C}.
\]

Thus, it is necessary and sufficient for \( A, B, \) and \( C \) to be complex numbers satisfying \( A^3 \neq -C^3 \), and \( A = B + C \) or \( A = -B \).