CC76. The point \( P(a, b) \) lies in the first quadrant. A line, drawn through \( P \), cuts the axes at \( Q \) and \( R \) such that the area of triangle \( OQR \) is \( 2ab \), where \( O \) is the origin. Prove that there are three such lines that satisfy these criteria.

*Originally 1996 Invitational Mathematics Challenge, Grade 11, problem 4.*

*We present the solution of Michel Bataille.*

Without loss of generality, we suppose that \( Q \) is on the \( x \)-axis and \( R \) on the \( y \)-axis. Then, \( Q(c, 0) \) and \( R(0, d) \) where \( c, d \) are nonzero real numbers. The equation of the line \( QR \) is \( \frac{x}{c} + \frac{y}{d} = 1 \) and \( P \) is on this line if and only if \( \frac{a}{c} + \frac{b}{d} = 1 \), that is, \( ad + bc = cd \).

The area of \( OQR \) being \( \frac{|c||d|}{2} \), the criterion about the area is satisfied if and only if \( |cd| = 4ab \), that is, \( cd = 4ab \) or \( cd = -4ab \).

Thus, setting \( u = \frac{a}{c} \) and \( v = \frac{b}{d} \), the criteria are satisfied if and only if \( u + v = 1 \) and \( \frac{1}{u} + \frac{1}{v} = 4 \) or \( \frac{1}{u} + \frac{1}{v} = -4 \).

Now, the first system formed by the equations \( u + v = 1 \) and \( \frac{1}{u} + \frac{1}{v} = 4 \) can be rewritten as \( u + v = 1 \) and \( uv = \frac{1}{4} \), whose unique solution is given by \( u = v = \frac{1}{2} \) (whence \( c = 2a, d = 2b \)).

The second system formed by the equations \( u + v = 1 \) and \( \frac{1}{u} + \frac{1}{v} = -4 \) can be rewritten as \( u + v = 1 \) and \( uv = -\frac{1}{4} \), which has two solutions for \( (u, v) \), namely \((\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2})\) and \((\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})\), whence \( (c, d) = (2(\sqrt{2} - 1)a, -2(\sqrt{2} + 1)b) \) or \( (c, d) = (-2(\sqrt{2} + 1)a, 2(\sqrt{2} - 1)b) \).

Altogether, three lines satisfy the criteria.

*Editor’s comments.* Note that it suffices to solve the problem for \( P(1, 1) \) since solutions for this problem correspond bijectively to those for the problem \( P(a, b) \) via the linear transformation \( \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \).

Some solvers considered three cases of solutions \( Q(c, 0) \) and \( R(0, d) \): (1) \( c > 0, d > 0 \), (2) \( c < 0, d > 0 \), and (3) \( c > 0, d < 0 \). If one reduces the problem to \( P(1, 1) \), one can show that there is a unique solution with \( c < 0, d > 0 \) if and only if there is a unique solution with \( c > 0, d < 0 \), since such solutions are in bijective correspondence via the reflection over the line \( y = x \).

CC77. The three following circles are tangent to each other: the first has radius \( a \), the second has radius \( b \), and the third has radius \( a + b \) for some \( a, b \in \mathbb{R} \) with \( a, b > 0 \). Find the radius of a fourth circle tangent to each of these three circles.
Inspired by the comments to question 10 of the 2013 Manitoba Mathematical Contest.

Several correct solutions and one incorrect solution were received. We present the solution of Neculai Stanciu and Titu Zvonaru.

Building a circle tangent to three circles is the problem of Apollonius. In the particular case in which the three given circles are tangent in pairs, the problem admits two solutions, known as the inner and outer Soddy circles. A formula for the radii of the Soddy circles in terms of the radii of the given circles can be found, for example, on http://mathworld.wolfram.com/SoddyCircles.html. It is

\[ r^2 = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 \pm \sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}. \]

In our case, \( r_1 = a \), \( r_2 = b \), and \( r_3 = a + b \).

This solves the problem if the three given circles are tangent externally. If, instead, the two smaller circles are interior to the larger circle, then the centers of the three given circles are collinear. Let \( O_1, O_2, O_3 \) be the centers of the circles with radii \( a, b, \) and \( a + b \), respectively. Let \( T \) be the center and \( r \) the radius of the circle tangent to these three. We apply Stewart’s Theorem to triangle \( TO_1 O_2 \) with cevian \( TO \), obtaining

\[ (OO_1)(TO_2)^2 + (OO_2)(TO_1)^2 = (O_1 O_2)(TO)^2 + (OO_1)(OO_2)^2 + (OO_2)(OO_1)^2. \]

Using \( OO_1 = b \), \( OO_2 = a \), \( O_1 O_2 = a + b \), \( TO_1 = r + a \), \( TO_2 = r + b \), and \( TO = a + b - r \), we have

\[ b(r + b)^2 + a(r + a)^2 = (a + b)(a + b - r)^2 + ba^2 + ab^2. \]

Solving for \( r \) gives

\[ r = \frac{ab(a + b)}{a^2 + b^2 + ab}. \]

**CC78.** Let \( g(x) = x^3 + px^2 + qx + r \), where \( p, q \) and \( r \) are integers. Prove that if \( g(0) \) and \( g(1) \) are both odd, then the equation \( g(x) = 0 \) cannot have three integer roots.

*Originally from 2001 Canadian Open Mathematics Challenge, problem B3b.*

We present two solutions. Two solvers followed the strategy of the first solution and the rest provided the second solution.

**Solution 1, by Šefket Arslanagić.**

We prove a stronger generalization:

Suppose that \( n \geq 2 \), \( g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + x_0 \) where each \( a_i \) is an integer, and \( g(0) \) and \( g(1) \) are both odd. Then \( g(x) \) does not have an integer root.

*Crux Mathematicorum, Vol. 40(6), June 2014*
Suppose otherwise that \( g(x) \) has an integer root \( n \). Then \( g(x) = (x - n)f(x) \) for some monic polynomial \( f(x) \) with integer coefficients. But then \( g(0) = -nf(0) \) and \( g(1) = -(n-1)f(1) \) cannot both be odd since one of \( n \) and \( n-1 \) is even. The result follows.

Solution 2, by Matei Coiculescu.

Since \( g(0) = r \) and \( g(1) = 1 + p + q + r \) are both odd, then \( r \) and \( p + q \) are both odd. Suppose that \( g(x) \) has three integer roots \( a, b, \) and \( c \). Then \( abc = -r \) is odd, so each of \( a, b, c \) is odd. Then so are \( p = -(a + b + c) \) and \( q = ab + bc + ca \), making \( p + q \) even. But this is a contradiction and the result follows.

CC79. Show that if \( n \) is an integer greater than 1, then \( n^4 + 4 \) is not prime.

Originally question 2 of 1979 APICS Math Competition.

We present the solution by Edward Wang.

The solution to this problem comes directly from a particular case of the well-known Sophie Germain identity:

\[
x^4 + 4y^4 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)
\]

Setting \( x = n \) and \( y = 1 \) yields \( n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2) \). Since \( n > 1 \), we have \( 1 < n^2 - 2n + 2 < n^2 + 2n + 2 \), so \( n^4 + 4 \) is a composite.

Editor’s comment. Edward Wang noticed that using similar arguments one can prove the following:

(i) For \( a, b \in \mathbb{N} \), \( a^4 + 4b^4 \) is a prime iff \( a = b = 1 \).

(ii) For \( n \in \mathbb{N} \), \( n^4 + 4^n \) is a prime iff \( n = 1 \).

CC80. Alphonse and Beryl play a game involving \( n \) safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the \( n \) keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects \( m \) of the safes (where \( m < n \)), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these \( m \) safes and tries to use these keys to open up the other \( n - m \) safes. If he can open a safe with one of the \( m \) keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let \( P_m(n) \) be the probability that Alphonse can eventually open all \( n \) safes starting from his initial selection of \( m \) keys. Determine a formula for \( P_2(n) \).


No solutions to this problem were received.