Sums of equal powers of natural numbers

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If someone asked you if you can calculate the sum of equal powers of consecutive natural numbers, you’d probably shrug and say sure, everyone can add. But there are many different ways to add. You probably know the story about young Gauss: while his classmates were labouring over calculating the sum 1 + 2 + \ldots + 100 term by term in order, he noticed that 1 + 100 = 2 + 99 = \ldots = 50+51 = 101 and quickly got the answer of 100·101/2 = 5050. [Ed.: Many authors describe the Gauss classroom story, sometimes with different details. Check out this article in American Scientist http://www.americanscientist.org/issues/pub/gausss-day-of-reckoning for more historical details.]

Of course computing sums of squares, cubes, fourth powers, and so on of natural numbers is quite a bit harder. In this article, we will consider three ways of constructing summation formulas that you can use to calculate any such sum. The sums we will be dealing with are of the form

\[ S_q(n) = 1^q + 2^q + \cdots + n^q, \]

where \( q \) is a nonnegative integer; so, for example

\[ S_2(4) = 1^2 + 2^2 + 3^2 + 4^2 = 30 \quad \text{and} \quad S_{10}(2) = 1^{10} + 2^{10} = 1025. \]

Clearly, \( S_0(n) = n \) for any \( n \in \mathbb{N} \).

1 What about the binomial theorem?

One technique we can use when calculating sums as in (1) is the binomial theorem. Consider it in the following form:

\[ (k-1)^{q+1} = k^{q+1} - \binom{q+1}{1} k^q + \binom{q+1}{2} k^{q-1} - \cdots + (-1)^q \binom{q+1}{q} k - (-1)^q. \] (2)

Let us now manipulate the sum (2) in the following way: move the first term of the right side to the left, let \( k = 1, 2, \ldots, n \), add all the resulting expressions and slightly massage them to get (using notations from (1))

\[ n^{q+1} = \binom{q+1}{1} S_q(n) - \binom{q+1}{2} S_{q-1}(n) + \cdots + (-1)^q \binom{q+1}{q} S_1(n) - (-1)^q S_0(n). \]

Solving this equation for \( S_q(n) \), we have:

\[ S_q(n) = \frac{1}{q+1} \left( n^{q+1} + \binom{q+1}{2} S_{q-1}(n) - \cdots - (-1)^q S_0(n) \right). \] (3)
Setting $q = 1$ gives $S_1(n) = \frac{1}{2} \left( n^2 + \binom{3}{2} S_0(n) \right) = \frac{1}{2} (n^2 + n)$. Now, substitute this expression for $S_1(n)$ as well as $q = 2$ into (3):

$$S_2(n) = \frac{1}{3} \left( n^3 + \binom{3}{2} S_1(n) - S_0(n) \right) = \frac{1}{3} \left( n^3 + \frac{3}{2} n^2 + \frac{1}{2} n \right).$$

Similarly, using (3), we can derive expressions for $S_3(n), S_4(n)$ and so on.

To experience the efficacy of these formulas, simply perform some calculations. For example, using the derived expression for $S_2(n)$, a complicated and clumsy sum $1^2 + 2^2 + \ldots + 100^2$ can be quickly computed as $S_2(100) = 338350$. Can you imagine how long it would take you to actually compute this sum term by term?

2 One term is better than $q$ terms

From Section 1 we see that you can use (3) to obtain the formulas for computing $S_q(n), q = 1, 2, \ldots$. Such a relation where each term of the sequence is defined using its preceding terms is called a recurrence relation. Clearly, the best recurrence relations are the ones that contain as few preceding terms as possible, preferably just the one immediately preceding the term you are computing. Luckily for us, we can find such a minimal recurrence relation for $S_q(n)$.

Let $P_m(n)$ be the following polynomial in terms of $n$ of degree $m$:

$$P_m(n) = a_0 n^m + a_1 n^{m-1} + \cdots + a_{m-1} n + a_m.$$  

Let $P_m^*(n)$ be the degree $m + 1$ polynomial that you get by replacing $n^k$ in $P_m(n)$ with the expressions

$$\frac{n^{k+1} - n}{k + 1}, \quad \text{for} \quad k = 0, 1, \ldots, m.$$  

The following relation is fairly straightforward: given two polynomials $P(n)$ and $Q(n)$,

$$(P(n) + Q(n))^* = P(n)^* + Q(n)^*. \quad (4)$$

**Theorem 1** The following two statements are true.

1. The sum $S_q(n)$ is a polynomial in terms of $n$ of degree $q + 1$ not containing the constant term.
2. The sum $S_q(n)$ can be defined by the recurrence

$$S_q(n) = n + S_{q-1}^*(n), \quad \text{with} \quad S_{-1} = 0. \quad (5)$$

**Proof.** We prove both assertions using mathematical induction.

1. Since $S_0(n) = n$, the recursion (5) is true for $q = 0$. Suppose it is also true for $S_0(n), S_1(n), \ldots, S_{q-1}(n)$. Then, plugging into the right-hand side of (3), we get a polynomial of degree $q + 1$ with no constant term. Therefore, the expression holds for $S_q(n)$ and by induction it is true for all $q \geq 0$.  

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2. Clearly, (5) holds for $q = 1$:

$$S_1(n) = n + n^* = n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}.$$ 

Suppose (5) holds for $q = 1, 2, \ldots, m - 1$. Substituting $q = m$ into (3), we get:

$$S_m(n) = \frac{1}{m+1} \left[ n^{m+1} + \binom{m+1}{2} n + (m-1)S^*_m(n) - \cdots - (-1)^m \binom{m+1}{m} n \right]$$

$$= \frac{1}{m+1} \left[ n^{m+1} + n \left( \binom{m+1}{2} - \binom{m+1}{3} + \cdots + (-1)^{m+1} \binom{m+1}{m} \right) \right]$$

$$+ (m-1) \binom{m+1}{2} S^*_m(n) - \cdots + (-1)^m \binom{m+1}{m} S^*_0(n).$$

Now note that since

$$1 - \binom{m+1}{1} + \binom{m+1}{2} - \cdots - (-1)^{m+1} \binom{m+1}{m} = 0,$$

we get

$$\binom{m+1}{2} - \binom{m+1}{3} + \cdots + (-1)^{m+1} \binom{m+1}{m} = \binom{m+1}{1} - 1 = m,$$

Also noting that

$$\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k},$$

and using (4), we finally get the following expression for $S_m(n)$:

$$S_m(n) = \frac{n^{m+1} + nm}{m+1} + \left[ \binom{m}{2} S_{m-2}(n) - \cdots - (-1)^m \binom{m}{m} S_0(n) \right]^*.$$

Setting $q = m - 1$ in (3), we see that the expression in square brackets above equals $mS_{m-1}(n) - n^m$ and thus

$$S_m(n) = \frac{n^{m+1} + nm + mS_{m-1}(n) - n^m}{m+1}$$

$$= \frac{n^{m+1} + nm}{m+1} + \frac{mS^*_m(n) - n^{m+1} - n}{m+1}$$

$$= n + mS^*_m(n).$$

Therefore the relation holds for $q = m$ and the proof is complete. ■

**Example 1** Given $S_4 = \frac{1}{4} (n^4 + 2n^3 + n^2)$ (check this), we can use Theorem 1 to find $S_4(n)$:

$$S_4(n) = n + 4S^*_4(n) = n + (n^4 + 2n^3 + n^2)^*$$

$$= n + \frac{n^5 - n}{5} + 2 \cdot \frac{n^4 - n}{4} + \frac{n^3 - n}{3} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$
3 The general formula

The binomial theorem can give us much more than (3). For example, consider the following transformations, where $m, n \in \mathbb{N}$:

$$m^n - 1 = (m - 1)^n - 1 = (m - 1) \binom{n}{1} + (m - 1)^2 \binom{n}{2} + \cdots + (m - 1)^n \binom{n}{n}.$$  

Suppose $m \neq 1$ and divide both sides by $m - 1$:

$$1 + m + m^2 + \ldots + m^{n-1} = \binom{n}{1} + (m - 1) \binom{n}{2} + \cdots + (m - 1)^{n-1} \binom{n}{n}. \quad (6)$$

**Problem 1** Prove that

$$a_0 + a_1 + \ldots + a_{n-1} = a_0 \binom{n}{1} + (a_1 - a_0) \binom{n}{2} + (a_2 - 2a_1 + a_0) \binom{n}{3} + \cdots$$

$$+ a_{n-1} - a_{n-2} \binom{n-1}{1} + a_{n-3} \binom{n-1}{2} - \cdots + a_0 (-1)^{n-1} \binom{n}{n}. \quad (7)$$

**Theorem 2** In the general case, we have the following formula for $S_q(n)$:

$$S_q(n) = \binom{n}{1} + (2^q - 1) \binom{n}{2} + (3^q - 2 \cdot 2^q + 1) \binom{n}{3} + \cdots$$

$$+ \left( q + 1 \right)^q - \left( \frac{q}{1} \right) q^q + \left( \frac{q}{2} \right) (q - 1)^q - \cdots + (-1)^q \binom{n}{q+1}. \quad (8)$$

**Sketch of the proof.** In (7), let $a_k = (k + 1)^q, k = 0, 1, \ldots, n - 1$ and first consider $n \leq q + 1$. Since in that case $\binom{n}{p} = 0$ for $p > q + 1$, (8) follows from (7). However, both sides of the expression are polynomials in $n$ of degree $q + 1$ with no constant terms, meaning they have to coincide for all $n$. ■

Formula (8) gives $S_q(n)$ as a linear combination of $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{q+1}$:

$$S_0(n) = \binom{n}{1}, \quad S_1(n) = \binom{n}{1} + \binom{n}{2},$$

$$S_2(n) = \binom{n}{1} + 3 \binom{n}{2} + 2 \binom{n}{3}, \quad S_3(n) = \binom{n}{1} + 7 \binom{n}{2} + 12 \binom{n}{3} + 6 \binom{n}{4},$$

and so on.

We now offer you, the reader, some exercises. Note that, for convenience, we use notation $S_q$ to denote $S_q(n)$.

**Problem 2** Show that

$$2S_1^2 = 2S_3,$$

$$4S_1^3 = 3S_5 + S_3,$$

$$8S_1^4 = 4S_7 + 4S_5,$$
and, in general, for \(k = 1, 2, \ldots\)

\[2^{k-1} S^k = \binom{k}{1} S_{2k-1} + \binom{k}{3} S_{2k-3} + \binom{k}{5} S_{2k-5} + \cdots\]

Moreover, the last term on the right-hand side of the above expression equals \(S_k\) or \(kS_{k-1}\) for odd and even \(k\), respectively.

**Problem 3** Show that for \(k = 1, 2, \ldots\)

\[3 \cdot 2^{k-1} S_2 S_1^{k-1} = \left(\binom{k}{0} + 2 \binom{k}{1}\right) S_{2k} + \left(\binom{k}{2} + 2 \binom{k}{3}\right) S_{2k-2} + \cdots\]

Moreover, the last term on the right-hand side of the above expression is equal to \((k+2)S_{k+1}\) or \(S_k\) for odd and even \(k\), respectively.

**Problem 4** Show that

\[S_3 = S_1^2,\]
\[S_5 = S_1^2 \cdot \frac{4S_1 - 1}{3},\]
\[S_7 = S_1^2 \cdot \frac{6S_1^2 - 4S_1 + 1}{3},\]

and, in general, \(S_{2k-1}\) (for \(2k - 1 \geq 3\)) is a polynomial in terms of \(S_1 = \frac{n(n+1)}{2}\) of degree \(k\) divisible by \(S_1^2\).

**Problem 5** Show that

\[S_4 = S_2 \cdot \frac{6S_1 - 1}{5},\]
\[S_6 = S_2 \cdot \frac{12S_1^2 - 6S_1 + 1}{7},\]
\[S_8 = S_2 \cdot \frac{40S_1^3 - 40S_1^2 + 18S_1 - 3}{15},\]

and, in general, \(S_{2k}/S_2\) is a polynomial in terms of \(S_1\) of degree \(k\).

**Problem 6** Prove that for \(k \geq 1\), we have \(S_k(-x - 1) = (-1)^{k-1} S_k(x)\).

**Problem 7** Find the sum \(1 + 27 + 125 + \cdots + (2n - 1)^3\).

**Problem 8** Find the sum \(2^2 + 5^2 + 8^2 + \cdots + (3n - 1)^2\).

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\ldots\]

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