No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge Edmund Swylan for the late correct solution to the problem 3761.

3841. Proposed by Marcel Chirita.

Let $ABC$ be a triangle with $a = BC$, $b = CA$, $c = AB$, $\angle A \leq \angle B \leq \angle C$ and $a^2 + b^2 = 2Rc$, where $R$ is the circumradius of $ABC$. Determine the measure of $\angle C$.

Many correct and three incorrect solutions were received.

Solution 1, by Cristóbal Sánchez-Rubio.

Let $O$ be the circumcircle of triangle $ABC$. Let $A'$ be the other end of the diameter through $A$ and let $B'$ be a point such that $AB' = \sqrt{a^2 + b^2}$. Then

$$a^2 + b^2 = 2Rc \iff AB' = \sqrt{a^2 + b^2} = \sqrt{2R \cdot c} \rightarrow c \leq AB' \leq 2R.$$  

But

$$C < 90^\circ \quad \Rightarrow \quad AB' > 2R > c$$  

$$C > 90^\circ \quad \Rightarrow \quad AB' < c < 2R.$$  

Both are impossible, so $C = 90^\circ$.

Solution 2, by the proposer Marcel Chirita.

From $a^2 + b^2 = 2Rc$, we have $4R^2 \sin^2 A + 4R^2 \sin^2 B = 4R^2 \sin C$, so that

$$\sin^2 A + \sin^2 B = \sin C.$$  

Rewriting gives successively

$$\sin^2 A + \sin^2 B = \sin(A + B),$$  

$$\sin^2 A + \sin^2 B = \sin A \cos B + \cos A \sin B,$$  

$$\sin A (\sin A - \cos B) = \sin B (\cos A - \sin B).$$  

Consider the following cases, noting that $A \leq B < 90^\circ$.

- If $0 < \cos B < \sin A$, then $0 < \sin B < \cos A$, and $1 = \cos^2 B + \sin^2 B < \sin^2 A + \cos^2 A = 1$, a contradiction.

- If $0 < \sin A < \cos B$, then $0 < \cos A < \sin B$, and $1 = \sin^2 A + \cos^2 A < \cos^2 B + \sin^2 B = 1$, a contradiction.

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• Otherwise, \( \sin A = \cos B \) and \( \cos A = \sin B \), implying that \( A + B = 90^\circ \) and \( C = 90^\circ \).

**Solution 3, by Omran Kouba.**

As in solution 2, we have \( \sin^2 A + \sin^2 B = \sin C \), which can be rewritten as

\[
2 \sin C = 2 - \cos(2A) - \cos(2B) = 2 - 2 \cos(A - B) \cos(A + B) = 2 + 2 \cos(A - B) \cos C,
\]

or equivalently \( \sin C = 1 + \cos(A - B) \cos C \geq 1 \). Hence \( C = \frac{\pi}{2} \).

3842. **Proposed by Jung In Lee.**

Let \( d(n) \) be the number of positive divisors of \( n \). For given positive integers \( a \) and \( b \), there exist infinitely many positive integers \( m \) such that \( d(a^m) \geq d(b^n) \); there also exist infinitely many positive integers \( n \) such that \( d(a^n) \leq d(b^m) \). Prove that \( d(a^k) = d(b^k) \) for any positive integer \( k \).

We present the solution by Joseph DiMuro.

Let \( a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \) and \( b = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s} \) be the prime factorizations of \( a \) and \( b \). Then for any positive integer \( k \) :

\[
d(a^k) = d(p_1^{ka_1} p_2^{ka_2} \cdots p_r^{ka_r}) = \prod_{i=1}^{r} (ka_i + 1), \quad \text{and} \quad \]

\[
d(b^k) = d(q_1^{kb_1} q_2^{kb_2} \cdots q_s^{kb_s}) = \prod_{j=1}^{s} (kb_j + 1). \]

Consider the polynomial

\[
f(x) = \prod_{i=1}^{r} (a_i x + 1) - \prod_{j=1}^{s} (b_j x + 1).
\]

For a positive integer \( x \), \( f(x) = d(a^x) - d(b^x) \). We were given that \( f(x) \geq 0 \) for infinitely many positive integer values of \( x \), and \( f(x) \leq 0 \) for infinitely many positive integer values of \( x \). Since \( f(x) \) is continuous, this implies that \( f(x) \) has infinitely many zeroes and is therefore the 0 polynomial. Thus, \( f(x) = 0 \) for all real values of \( x \), so \( d(a^k) = d(b^k) \) for any positive integer \( k \).

3843. **Proposed by George Apostolopoulos.**

Let \( a, b \) be distinct real numbers such that

\[
a^4 + b^4 - 3(a^2 + b^2) + 8 \leq 2(a + b)(2 - ab).
\]

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Find the value of the expression
\[ A = (ab)^n + (ab + 1)^n + (ab + 2)^n, \]
where \( n \) is a positive integer.

*We present the solution by Salem Malikić and Nermin Hodžić (done independently).*

The given condition is equivalent to
\[ 2(a^4 + b^4 - 3(a^2 + b^2) + 8 - 2(a + b)(2 - ab)) \leq 0, \]
which may be rewritten as
\[ (a - b)^2(a + b - 1)^2 + (a^2 + b^2 + a + b - 4)^2 \leq 0. \]

This implies that \( a \) and \( b \) must satisfy the two equations
\[ a^2 + b^2 + a + b - 4 = 0, \quad (a - b)(a + b - 1) = 0. \]

Since \( a \) and \( b \) are distinct, the last equation implies that \( a + b - 1 = 0 \), so that \( a + b = 1 \), and the first equation becomes \( a^2 + b^2 = 3 \). Then
\[ ab = \frac{(a + b)^2 - (a^2 + b^2)}{2} = -1, \]
so that
\[ A = (ab)^n + (ab + 1)^n + (ab + 2)^n = (-1)^n + 1 = \begin{cases} 0 & n \text{ odd}, \\ 2 & n \text{ even}. \end{cases} \]

*Editor’s comments :* There were two main solution types. The first was to rearrange inequalities and equations to solve for \( a \) and \( b \), similar to the above; the second was to use calculus to find \( a \) and \( b \) (as they are critical points of the polynomial that is being set less than 0). Malikić and Hodžić used the first solution method, but instead of solving for \( a \) and \( b \), they simply solved for \( ab \), which saves some work.

**3844. Proposed by Michel Bataille.**

Find the intersection of the surface with equation
\[ (x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2 = (x + y)(y + z)(z + x) \]
with the plane \( x + y + z = 2 \).

*Among all the received solutions, one was incorrect. We present three solutions.*

*Solution 1, by Nermin Hodžić.*
We have
\[(x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2 \geq (1) \frac{3}{4}(x^2 + y^2 + z^2 + x^2)^2 = \frac{4(x^2 + y^2 + z^2)^2}{3}
\geq (2) \frac{4(|x| + |y| + |z|)}{27}
\geq (3) \frac{8(|x| + |y| + |z|)^3}{27}
\geq (4) \frac{|x+y| + |y+z| + |z+x|)^3}{27}
\geq (5) |(x+y)(y+z)(z+x)|
\geq (x+y)(y+z)(z+x),
\]
with equality if and only if \(x = y = z = \frac{2}{3}\). Here (1) follows from Cauchy’s inequality applied to the vectors < \(x^2 + y^2, y^2 + z^2, z^2 + x^2 > \) and < 1, 1, 1 >; (2) is the AM-QM inequality; (3) follows from the hypothesis \(x + y + z = 2\); (4) follows from the triangle inequality, and (5) follows from the AM-GM inequality.

\textit{Solution 2, by Omran Kouba, modified slightly by the editor.}

By the AM-QM inequality, \(2(x^2 + y^2) \geq (|x| + |y|)^2\), and by Muirhead’s inequality, \(a^4 + b^4 + c^4 \geq (a + b + c)abc\) for any nonnegative real numbers \(a, b, c\), so that
\[4((x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2)
\geq (1) (|x| + |y|)^4 + (|y| + |z|)^4 + (|z| + |x|)^4
\geq (2) 2(|x| + |y| + |z|)(|x| + |y|)(|y| + |z|)(|z| + |x|)
\geq (3) 2|x + y + z||x + y||y + z||z + x|,
\]
with equality if and only if \(x = y = z\). Hence for \(x + y + z = 2\), we see that
\[(x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2 \geq |(x+y)(y+z)(z+x)|,
\]
with equality if and only if \(x = y = z = \frac{2}{3}\). This proves that \((\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\) is the unique point of intersection.

\textit{Solution 3, composite of similar solutions by Salem Malikić and Šefket Arslanagić.}

At a point of intersection,
\[2[(x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2] - (x + y + z)(x + y + (y + z)(z + x) = 0,
\]
which is successively equivalent to
\[4(x^4 + y^4 + z^4) + 2(x^2 y^2 + y^2 z^2 + z^2 x^2) - 4(x^2 y z + z x y z^2) - (x^3 y + xy^3 + y^3 z + y z^3 + z^3 x + z x^3) = 0.
\]

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Now,
\[
\begin{align*}
(x^2 - xy)^2 + (x^2 - xz)^2 + (y^2 - yz)^2 + (y^2 - zx)^2 + (z^2 - zy)^2 \\
+ (x^2 - yz)^2 + (y^2 - zx)^2 + (z^2 - xy)^2 + (y^2 - yz)^2 + (z^2 - x^2)^2 \\
+ (xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2
\end{align*}
\]
For this to hold, each term must equal 0, which in turn implies that \( x = y = z = \frac{2}{3} \). Since \( \left( \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right) \) is readily verified to be a solution, it is the unique solution.

3845. Proposed by Dao Thanh Oai.

Let the six points \( A_1, A_2, \ldots, A_6 \) lie in that order on a circle, and the six points \( B_1, B_2, \ldots, B_6 \) lie in that order on another circle. If the quadruples \( A_i, A_{i+1}, B_{i+1}, B_i \) lie on circles with centres \( C_i \) for \( i = 1, 2, \ldots, 5 \), then prove that \( A_6, A_1, B_1, B_6 \) must also lie on a circle. Furthermore, if \( C_6 \) is the centre of the new circle, then prove that lines \( C_1 C_4, C_2 C_5, \) and \( C_3 C_6 \) are concurrent.

No solutions to this problem were received. We present a solution by a member of Editorial Board (J. Chris Fisher). We pose one part of the solution as problem 3945.

The order of the points on their respective circles is not important (except for making a nice picture):

The first claim follows immediately from two applications of Miquel’s theorem, which says that a chain of three circles will close with a fourth circle; more precisely, in the statement of our problem restricted to the quadruples \( A_i, A_{i+1}, B_{i+1}, B_i \) lying on circles with \( i \) running from 1 to 3, the theorem states that the points \( A_1, B_1, A_4, A_4 \) must lie on a circle. Now we have a second chain of three circles, namely the new circle \( A_1 B_1 B_4 A_4 \) with the remaining two circles \( A_4 A_5 B_5 B_4 \) and \( A_5 A_6 B_6 B_5 \), whence (by Miquel’s theorem) the remaining four points \( A_6, A_1, B_1, B_6 \) will also lie on a circle, as required. A similar argument will work for any chain.
having an even number of circles (with $2k$ points on both the $A$ and the $B$ circles, and $i$ running from 1 to $2k - 1$): the points $A_{2k}, A_1, B_1, B_{2k}$ must also lie on a circle.

For the second claim we will see that the sides of the hexagon $C_1C_2C_3C_4C_5C_6$ are tangent to a conic whose foci are the centres, call them $A$ and $B$, of the circles $A_i$ and $B_i$; by Brianchon’s theorem, the lines joining opposite vertices of the hexagon, namely $C_1C_4, C_2C_5,$ and $C_3C_6$, must be concurrent. To this end, we wish to show that for each $i$, the unique conic with foci $A$ and $B$ that is tangent to the line $C_iC_{i-1}$ (joining the centres of consecutive circles) coincides with the unique conic with those foci that is tangent to the line $C_iC_{i+1}$. Note that the conic will be an ellipse if the tangent $C_iC_{i+1}$ misses the line segment $AB$; it is a hyperbola if the tangent intersects $AB$ between $A$ and $B$. (To avoid the line passing through $A$ or $B$ we should insist that none of the $A_i$ lie on the circle containing the $B_i$, and vice versa.) The second claim thereby reduces to a theorem that seems as if it should have been known a century ago, for which it seems to be easier to find a proof than a reference. The editor J. Chris Fisher now poses this proof as problem 3945, which appears in this issue of Crux.

3846. Proposed by Arkady Alt.

Let $r$ be a positive real number. Prove that the inequality

$$\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} + \frac{1}{1 + c + c^2} \geq \frac{3}{1 + r + r^2}$$

holds for any positive $a, b, c$ such that $abc = r^3$ if and only if $r \geq 1$.

We present the proof by the proposer, modified and expanded by the editor.

We first prove the following lemma:

**Lemma.** Let $r$ be a given positive number. Then

$$\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} \geq \frac{2}{1 + r + r^2} \quad (1)$$

for any positive $a$ and $b$ with $ab = r^2$ if and only if $r \geq r_0$, where $r_0$ is the unique positive root of the equation $4x^3 + 3x^2 - 3x - 1 = 0$.

[Editor: Let $f(x) = 4x^3 + 3x^2 - 3x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 3 > 0$, so $f$ has a real root $r_0 \in (0, 1)$. By Rule of signs, $r_0$ is the only positive root.]

**Proof.** Note that if (1) holds for any positive $a$ and $b$ with $ab = r^2$, then

$$\lim_{a \to \infty} \left( \frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} \right) = 1 \geq \frac{2}{1 + r + r^2}$$

if and only if $r^2 + r - 1 \geq 0$, so $r \geq \frac{\sqrt{5} - 1}{2}$.

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Now, suppose \( a, b > 0 \) with \( ab = r^2 \), where \( r \geq \frac{\sqrt{5} - 1}{2} \). Let \( x = a + b \), then \( x \geq 2\sqrt{ab} = 2r \), and

\[
\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} - \frac{2}{1 + r + r^2} = \frac{2 + a + b + a^2 + b^2}{1 + a + b + ab + ab(a + b) + a^2 + b^2 + a^2b^2} - \frac{2}{1 + r + r^2} = \frac{2 + x + x^2 - 2r^2}{1 + x + r^2 + (r^2)x + x^2 - 2(r^2) + r^4} - \frac{2}{1 + r + r^2} = \frac{x^2 + x + 2 - 2r^2}{x^2 + r^2x + x - r^2 + r^4 + 1} - \frac{2}{1 + r + r^2} = \frac{P(x)}{Q(x)},
\]

where \( Q(x) = (x^2 + r^2x + x - r^2 + r^4 + 1)(1 + r + r^2) \) and by tedious computations together with synthetic division, we have:

\[
P(x) = (x^2 + x + 2 - 2r^2)(1 + r + r^2) - 2(x^2 + r^2x + x - r^2 + r^4 + 1) = (r^2 + r - 1)x^2 + (-r^2 + r - 1)x + 2(1 - r^2)(r^2 + r + 1) - 2(1 - r^2 + r^4) = (r^2 + r - 1)x^2 - (r^2 - r + 1)x - 4r^4 - 2r^3 + 2r^2 + r = (x - 2r)((r^2 + r - 1)x + 2r^3 + r^2 - r - 1).
\]

Since \( x \geq 2r \) and clearly \( Q(x) > 0 \), we have \( \frac{P(x)}{Q(x)} \geq 0 \) if and only if

\[
(r^2 + r - 1)x + 2r^3 + r^2 - r - 1 \geq 0. \tag{2}
\]

Since \( x \geq 2r \) and \( r^2 + r - 1 \geq 0 \), (2) holds if and only if it holds for \( x = 2r \); that is, \( 2r(r^2 + r - 1) + 2r^3 + r^2 - r - 1 \geq 0 \) or \( 4r^3 + 3r^2 - 3r - 1 \geq 0 \) or \( 4r^2 + 3r - 3 - \frac{1}{r} \geq 0 \).

The function \( g(r) = 4r^2 + 3r - 3 - \frac{1}{r} \) is increasing on \((0, \infty)\) and \( g(\frac{1}{2}) = -\frac{5}{2} < 0, g(\frac{1}{2}) = \frac{5}{2} > 0 \), so it has only one root \( r_0 \) and \( r_0 \in (\frac{1}{2}, \frac{3}{2}) \). Hence, \( r_0 \) is the smallest value of \( r \) such that (2) holds for all \( x \geq 2r \).

Furthermore, if we set \( r_1 = \frac{\sqrt{5} - 1}{2} \), then

\[
4r_1^3 + 3r_1^2 - 3r_1 - 1 = 4r_1(r_1^2 + r_1 - 1) - r^2 + r_1 - 1 + 2(r_1 - 1) = 2(r_1 - 1) = \sqrt{5} - 3 < 0,
\]

so \( r_1 < r_0 < \frac{3}{4} \).

In particular, (1) holds for all \( a, b > 0 \) such that \( ab = r^2 \) if \( r \geq 1 \) and this completes the proof of the lemma.

Using this lemma, we now prove that for all \( a, b, c > 0 \) with \( abc = r^3 \),

\[
\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} + \frac{1}{1 + c + c^2} \geq \frac{3}{1 + r + r^2}
\]

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if and only if \( r \geq 1 \).

**Necessity.** Setting \( c = \frac{r^3}{n} \) and \( a = b = n \) in the given inequality where \( n \) is an arbitrary natural number, and passing to the limit, we have

\[
1 = \lim_{n \to \infty} \left( \frac{2}{1 + n + n^2} + \frac{1}{1 + \frac{r^3}{n} + \frac{r^6}{n^2}} \right) \geq \frac{3}{1 + r + r^2},
\]

which implies \( r^2 + r - 2 \geq 0 \) or \( (r + 2)(r - 1) \geq 0 \), so \( r \geq 1 \).

**Sufficiency.** Let \( a, b, c > 0 \) with \( abc = r^3, r \geq 1 \). Without loss of generality, assume that \( a \geq b \geq c \). Then \( c^3 \geq abc = r^3 \), so \( c \geq r \). Set \( x = \sqrt{ab} \). Then \( c = \frac{r^3}{x^2} \) and \( x \geq c \), so \( x \geq r \). Since \( ab = x^2 \), we have, by the lemma, that

\[
\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} \geq \frac{2}{1 + x + x^2}.
\]

Hence, it suffices to prove that for any \( x \) and \( r \) with \( x \geq r \geq 1 \), we have

\[
\frac{2}{1 + x + x^2} + \frac{1}{1 + \frac{r^3}{x^2} + \frac{r^6}{x^4}} \geq \frac{3}{1 + r + r^2}.
\]

Let

\[
D(x) = \frac{2}{1 + x + x^2} + \frac{1}{1 + \frac{r^3}{x^2} + \frac{r^6}{x^4}} - \frac{3}{1 + r + r^2}.
\]

Then

\[
D(x) = \frac{1}{1 + \frac{r^3}{x^2} + \frac{r^6}{x^4}} - \frac{1}{1 + r + r^2} - 2 \left( \frac{1}{1 + r + r^2} - \frac{1}{1 + x + x^2} \right)
\]

\[
= \frac{r + r^2 - \frac{r^3}{x^2} - \frac{r^6}{x^4}}{(1 + \frac{r^3}{x^2} + \frac{r^6}{x^4})(1 + r + r^2)} - \frac{x + x^2 - r - r^2}{(1 + r + r^2)(1 + x + x^2)} = A(x)
\]

where

\[
A(x) = \frac{x^2r(x^2 - r^2) + r^2(x^4 - r^4)}{x^4 + x^2r^3 + r^6} - \frac{2(1 + x + r)(x - r)}{1 + x + x^2}
\]

\[
= \frac{(x^2 - r^2)(rx^2 + r^2(x^2 + r^2))(x^2 + x + 1) - 2(x + r + 1)(x - r)(x^4 + r^3x^2 + r^6)}{(x^4 + x^2r^3 + r^6)(x^2 + x + 1)}
\]

\[
= \frac{(x - r)B(x)}{(x^4 + x^2r^3 + r^6)(x^2 + x + 1)},
\]

where

\[
B(x) = (x + r)(x^2 + x + 1)(rx^2 + r^2(x^2 + r^2)) - 2(x + r + 1)(x^4 + r^3x^2 + r^6)
\]

\[
= (rx^2 + r^2(x^2 + r^2))(x^2 + x + 1) - 2(x + r + 1)(x^4 + r^3x^2 + r^6)
\]

\[
= (r^2 + r - 2)x^5 + (r^3 + 2r^2 - r - 2)x^4 + (r^4 - r^3 + 2r^2 + r)x^3
\]

\[
+ (r^5 - r^4 - r^3 + r^2)x^2 + (-2r^6 + r^5 + r^4)x - 2r^7 - 2r^6 + r^5
\]

\[
= (x - r)E(x),
\]

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\[ E(x) = (r^2 + r - 2)x^4 + (2r^3 + 3r^2 - 3r - 2)x^3 + (3r^4 + 2r^3 - r^2 - r)x^2 \\
+ (4r^5 + r^4 - 2r^3)x + (2r^6 + 2r^5 - r^4). \]

It then suffices to prove that \( E(x) \geq 0 \) for all \( x \geq r \). Since
\[ E(r) = r^6 + r^5 - 2r^4 + 2r^6 + 3r^5 - 3r^4 - 2r^3 + 3r^6 + 2r^5 - r^3 \\
+ 4r^6 + r^5 - 2r^4 + 2r^6 - 2r^5 - r^4 \\
= r^7 + 12r^6 + 8r^5 - 9r^4 - 4r^3 \\
= r^7 + 3r^6 + 4r^5 + 3(r - 1)(9r + 4)(r + 1) \geq 0 \]
for \( r \geq 1 \) and since all the coefficients of \( E(x) \) are clearly nonnegative as well, we conclude that \( E(x) \geq E(r) > 0 \) for all \( x \geq r \). Hence, \( B(x) \geq 0 \) and from (8) \( A(x) \geq 0 \) and finally from (5) \( D(x) \geq 0 \), which establishes (3) and completes the proof.

*Editor’s comment.* Perfetti’s solution was computer assisted and Pranesachar’s solution used Maple.

**3847. Proposed by Jung In Lee.**

Prove that there are no distinct positive integers \( a, b, c \) and nonnegative integer \( k \) that satisfy the conditions
\[ a^{b+k} | b^{a+k}, \quad b^{c+k} | c^{b+k}, \quad c^{a+k} | a^{c+k}. \]

We present the solution by Joseph DiMuro.

We prove the stronger result that there are no distinct positive integers \( a, b, c \) and nonnegative real number \( k \) that satisfy the conditions
\[ a^{b+k} \leq b^{a+k}, \quad b^{c+k} \leq c^{b+k}, \quad c^{a+k} \leq a^{c+k}. \quad (1) \]

Suppose (1) holds. Then from \( a^{b+k} \leq b^{a+k} \), we have
\[ \ln(a^{b+k}) \leq \ln(b^{a+k}) \quad \text{or} \quad (b + k) \ln a \leq (a + k) \ln b, \]
so
\[ \frac{\ln a}{a + k} \leq \frac{\ln b}{b + k}. \]

Similarly, from the other inequalities in (1), we deduce that
\[ \frac{\ln b}{b + k} \leq \frac{\ln c}{c + k} \quad \text{and} \quad \frac{\ln c}{c + k} \leq \frac{\ln a}{a + k}. \]
Therefore, we have
\[
\ln a = \frac{\ln b}{b+k} = \frac{\ln c}{c+k}.
\]
Now, let \( f_k(x) = \frac{\ln x}{x+k}, \quad x \in (0, \infty) \). Then \( f_k(x) \) is a continuous function such that
\[
f_k(a) = f_k(b) = f_k(c)
\]
and it suffices to show that \( (2) \) cannot hold. Since
\[
\frac{1}{x+k} - \ln x = \frac{1 + \frac{k}{x} - \ln x}{(x+k)^2},
\]
we have \( f'_k(x) = 0 \) if and only if \( 1 + \frac{k}{x} - \ln x = 0 \). Let \( g(x) = 1 + \frac{k}{x} - \ln x \). Then \( g'(x) = -\frac{k}{x^2} - \frac{1}{x} < 0 \), so \( g(x) \) is a strictly decreasing function. Since
\[
\lim_{x \to 0^+} g(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) = -\infty,
\]
we see that \( g(x) = 0 \) for exactly one value of \( x \), so \( f_k(x) \) has exactly one critical value. Hence, \( (2) \) cannot hold and our proof is complete.

**3848. Proposed by Rudolf Fritsch.**

We define an altitude of the plane \((2n+1)\)-gon \( A_0A_1\ldots A_{2n} \) to be the line through vertex \( A_i \) perpendicular to the opposite side \( A_i-nA_i+n \) (where indices are reduced modulo \( 2n+1 \)). Prove that if \( 2n \) of the altitudes are concurrent, then the remaining altitude passes through the point of concurrence.

We present a composite of the similar solutions by Oliver Geupel and by Günter Pickert.

For an arbitrary set of \( 2n+1 \) vectors \( \vec{A}_i \), whose indices \( -n \leq i \leq n \) are considered to be integers modulo \( 2n+1 \) (or, more precisely, to be the elements of the factor group \( \mathbb{Z}_{2n+1} = \mathbb{Z}/(2n+1)\mathbb{Z} \)), one has
\[
\sum_{-n}^{n} \vec{A}_i \cdot \vec{A}_{i-n} = \sum_{-n}^{n} \vec{A}_{j+n} \cdot \vec{A}_j = \sum_{-n}^{n} \vec{A}_i \cdot \vec{A}_{i+n},
\]
and thus
\[
\sum_{-n}^{n} \vec{A}_i \cdot (\vec{A}_{i+n} - \vec{A}_{i-n}) = 0. \tag{1}
\]

We now turn to the problem. Assume that the altitudes from \( A_1, A_2, \ldots, A_{2n} \) are concurrent in the point \( O \) and consider position vectors relative to the origin \( O \). We are given that for \( -n \leq i \leq n, \ i \neq 0 \), the line \( A_iO \) is perpendicular to the side \( A_{i+n}A_{i-n} \), which is expressed by the equation
\[
\vec{A}_i \cdot (\vec{A}_{i+n} - \vec{A}_{i-n}) = 0.
\]
That is, 2n of the summands in the sum (1) vanish, and we conclude that the remaining summand must also vanish, namely
\[ \vec{A}_0 \cdot (\vec{A}_{-n} - \vec{A}_n) = 0. \]
Thus the line OA0 is perpendicular to the side A−nA_n, which proves that the altitude from the vertex A0 also passes through the point O.

Editor’s comments. The proposer observed that the result holds for a quite general definition of polygon: the proof makes clear that its vertices can be any ordered set of 2n + 1 points, not necessarily distinct, that satisfy the hypothesis.

Professor Pickert died on February 11, 2015, a few months before his 98th birthday. He was active until the end, as can be seen in the above solution and in the three-part article (co-authored by Rudolf Fritsch) that appeared last year in Volume 39 of Crux.

3849. Proposed by José Luis Díaz-Barrero.

Let \( A(z) = \sum_{k=0}^{n} a_k z^k \) be a polynomial of degree n with complex coefficients having all its zeros in the disc \( \mathcal{C} = \{ z \in \mathbb{C} : |z| \leq \sqrt{6} \} \). Show that
\[ |A(3z)| \geq \left( \frac{2\sqrt{2}}{3} \right)^{n/2} |A(2z)| \]
for any complex number \( z \) with \( |z| = 1 \).

We present a mixture of the solutions by Omran Kouba and Alexander Mangerel.

We prove a stronger statement than the given problem, namely the above problem with \( \frac{2}{3} \) replaced with \( \frac{3}{2} \). To do this, we use the following claim.

Claim: For \( w \in \mathbb{C} \), \( |3z - w| \geq \sqrt{\frac{3}{2}} |2z - w| \) for any \( z \in \mathbb{C} \) with \( |z| = 1 \) iff \( |w| \leq \sqrt{6} \).

To see this, square both sides of the first inequality, rewrite using conjugates, and expand to obtain
\[ 9 - 3(\bar{z}w + z\bar{w}) + |w|^2 \geq \frac{3}{2} (4 - 2(\bar{z}w + z\bar{w}) + |w|^2). \]
Upon simplification, this yields \(|w|^2 \leq 6\), and taking square roots proves the claim, as we are dealing with positive quantities whenever square roots are taken.

The proof of the main statement follows by writing \( A(z) = a_n \prod_{i=1}^{n} (z - w_i) \) where \( w_i \) are the roots of \( A \), letting \(|z| = 1\), and taking absolute values, using the fact that each root of \( A \) has \(|w| \leq \sqrt{6}\):
\[ |A(3z)| = |a_n| \prod_{i=1}^{n} |3z - w_i| \geq |a_n| \prod_{i=1}^{n} \sqrt{\frac{3}{2}} |2z - w_i| = \left( \frac{3}{2} \right)^{\frac{n}{2}} |A(2z)|. \]
Editor’s comments. Mangerel noted that he believed that the intent of the problem was to have the fraction flipped, and proved the tighter bound via slightly different calculations. The specific problem may be abstracted to the following:

Let \( a \geq b \in \mathbb{R} \), and let \( A(z) \) be a polynomial with complex coefficients such that all of its roots are in the contained in the disk with radius \( \sqrt{ab} \). Show that

\[
|A(az)| \geq \left( \frac{a}{b} \right)^2 |A(bz)|
\]

for any complex \( z \) with \( |z| = 1 \).

The proof is the same as above when \( a \neq b \), with \( a \) replacing 3 and \( b \) replacing 2, and the case where \( a = b \) is obviously trivial.

3850. Proposed by Lee Sallows and Stan Wagon.

Each of the four networks shown uses the same four distinct integer-valued resistors \( a, b, c, d \), and the total resistances of the networks themselves are again \( a, b, c, d \). Find values of \( a, b, c, d \) that work.

We present a summary of the solutions by Richard Hess and the proposer.

Richard Hess found the solution \((a, b, c, d) = (k, 2k, 3k, 4k)\) which for the four resistors gives

\[
R_1 = \frac{1}{\frac{1}{2k} + \frac{1}{4k} + \frac{1}{k+3k}} = k, \\
R_2 = \frac{1}{\frac{1}{2k} + \frac{1}{3k} + \frac{1}{k} + \frac{1}{4k}} = 2k, \\
R_3 = \frac{1}{\frac{1}{4k} + \frac{1}{k+3k}} + 2k = 4k, \\
R_4 = \frac{1}{\frac{1}{3k} + \frac{1}{2k+4k}} + k = 3k.
\]

The proposers used an exhaustive computer search which confirms that this is the only solution.

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