OLYMPIAD SOLUTIONS

OC121. Prove that for all positive real numbers $x, y, z$ we have
\[
\sum_{\text{cyc}} (x + y)\sqrt{(z + x)(z + y)} \geq 4(xy + yz + zx).
\]

Originally question 2 from the 2012 Balkan Mathematical Olympiad.

We present two solutions.

Solution 1, composed of similar solution of David Manes and Paolo Perfetti.

By Cauchy-Schwarz we have 
\[
(z + x)(z + y) \geq \left(\sqrt{z}\sqrt{x} + \sqrt{x}\sqrt{y}\right)^2.
\]
Moreover, AM-GM gives 
\[
x + y \geq 2\sqrt{xy}.
\]
Thus
\[
\sum_{\text{cyc}} (x + y)\sqrt{(z + x)(z + y)} \geq \sum_{\text{cyc}} (x + y)(z + \sqrt{xy})
\]
\[
= \sum_{\text{cyc}} (x + y)z + \sum_{\text{cyc}} (x + y)\sqrt{xy}
\]
\[
\geq \sum_{\text{cyc}} xz + yz + 2\sum_{\text{cyc}} xy
\]
\[
= 4(xy + yz + zx).
\]

Solution 2, composed of similar solutions by Arkady Alt and Šefket Arslanagić.

Let 
\[
a := \sqrt{y + z}, \quad b := \sqrt{z + x}, \quad c := \sqrt{x + y}.
\]

Then $a, b, c$ are the side lengths of an acute triangle, because 
\[
\frac{b^2 + c^2 - a^2}{2} = x > 0, \quad \frac{c^2 + a^2 - b^2}{2} = y > 0, \quad \frac{a^2 + b^2 - c^2}{2} = z > 0.
\]

Moreover, we have 
\[
4(xy + yz + zx) = \sum_{\text{cyclic}} (b^2 + c^2 - a^2) \left(\frac{c^2 + a^2 - b^2}{2}\right)
\]
\[
= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 16F^2,
\]
where $F$ is the area of the triangle.

Let $R, r, s$ be circumradius, inradius and semiperimeter of the triangle. Then, original inequality becomes 
\[
abc(a + b + c) \geq 16F^2 \iff 8FRs \geq 16F^2 \iff Rs \geq 2F \iff Rs \geq 2sr \iff R \geq 2r,
\]
where latter inequality is the well known Euler’s Inequality.

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We define a sequence $f_n(x)$ of functions by

$$f_0(x) = 1, f_1(x) = x, \quad (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x), \text{ for } n \geq 1.$$ 

Prove that for every $n$, $f_n(x)$ is a polynomial with integer coefficients.

*Originally question 3 from the Indian national Olympiad 2012.*

We give the solution of Omran Kouba.

We first prove by induction on $n$ that for all $\theta \not\in \mathbb{Q}\pi$, we have

$$f_n(2 \cos \theta) = \frac{\sin((n + 1)\theta)}{\sin \theta}. \quad (1)$$

This is trivially true for $n = 0, 1$. So, let us suppose that this is true for $n$ and $n - 1$, then for $n\theta \not\in \pi\mathbb{Z}$ we have

$$f_{n+1}(2 \cos \theta) = \frac{f_n^2(2 \cos \theta) - 1}{f_{n-1}(2 \cos \theta)} = \frac{\sin^2((n + 1)\theta) - \sin^2 \theta}{\sin \theta \sin(n\theta)} = \frac{\cos(2\theta) - \cos(2(n + 1)\theta)}{2 \sin \theta \sin(n\theta)}$$

$$= \frac{2 \sin((n + 2)\theta) \sin(n\theta)}{2 \sin \theta \sin(n\theta)} = \frac{\sin((n + 2)\theta)}{\sin \theta}.$$

Now for a given $n \geq 1$, we define the function

$$Q_n(x) = f_{n+1}(x) + f_{n-1}(x) - xf_n(x).$$

It is easy to see that all $f_n$ are rational functions, and hence so are $Q_n$. Using (1), we see that $Q_n(2 \cos \theta) = 0 \quad \forall \theta \not\in \mathbb{Q}\pi$. Therefore, $Q_n$ is a rational function which has infinitely many zeroes. Thus $Q_n \equiv 0$, which yields

$$f_{n+1}(x) = xf_n(x) - f_{n-1}(x).$$

As $f_0 = 1, f_1 = x$, by induction it follows immediately that $f_n \in \mathbb{Z}[X]$.

*OC123.* Let $p$ be prime. Find all positive integers $n$ for which, whenever $x$ is an integer such that $x^n - 1$ is divisible by $p$, then $x^n - 1$ is also divisible by $p^2$.

*Originally question 3 from Japan Math Olympiad 2012.*

No solution was received to this problem. We give a solution by the editor.

We claim that $n$ has the desired property if and only if $p|n$.

$\Rightarrow$ Since $p|(1 + p)^n - 1$ it follows that $p^2((1 + p)^n - 1$. Therefore

$$0 \equiv (1 + p)^n - 1 \equiv np \pmod{p^2}.$$

This shows that $p|n$.

$\Leftarrow$ This implication is an immediate consequence of the Hensel’s Lemma:

If $f(X) = X^n - 1$, then as $f(x) \equiv 0 \pmod{p}$ and $f'(x) \equiv 0 \pmod{p}$, it follows that $f(x) \equiv 0 \pmod{p^2}$.
We provide below a more elementary solution. Let \( n = pk \). Then
\[
x^n - 1 \equiv (x^p)^k - 1 \equiv x^k - 1 \pmod{p}.
\]

Let \( y := x^k \). Then we know that \( y \equiv 1 \pmod{p} \), and hence
\[
1 + y + y^2 + \cdots + y^{p-1} \equiv 1 + 1 + \cdots + 1 \equiv 0 \pmod{p}.
\]

This shows that \( p | 1 + y + y^2 + \cdots + y^{p-1} \). As \( p \) also divides \( 1 - y \), we get that
\[
p^2 | 1 - y^p = 1 - x^n.
\]


**OC124.** Find all triples \((a, b, c)\) of positive integers with the following property: for every prime \( p \), if \( n \) is a quadratic residue \((\mod p)\), then \( an^2 + bn + c \) is also a quadratic residue \((\mod p)\).

Originally question 2 from the 2012 Romanian Team Selection Test, day 5.

We present the solution by Oliver Geupel.

The triples
\[
(a, b, c) = (u^2, 2uw, w^2)
\]
with \( u, w \in \mathbb{N} \) have the desired property, because \( an^2 + bn + c = (un + w)^2 \) is a perfect square and therefore a quadratic residue modulo every prime. We will show that there are no other solutions.

Let \( a, b, c \) be natural numbers with the desired property and consider the polynomial
\[
P(x) = ax^4 + bx^2 + c.
\]

Then, for every natural number \( n \), the number \( P(n) \) is a quadratic residue modulo every prime by hypothesis.

By a well-known application of Chebotarev’s density theorem, an integer is a perfect square if it is a quadratic residue modulo every prime. Hence, \( P(n) \) is a perfect square for every natural number \( n \).

Let \( P(x) = (F(x))^2G(x) \) with polynomials \( F, G \in \mathbb{Z}[x] \) where the polynomial
\[
G(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0
\]
is square-free in \( \mathbb{Z}[x] \). The degree of \( G \) is either 0 or 2 or 4. We prove by contradiction that \( \deg G = 0 \).

Assume that \( \deg G \geq 2 \). Because \( G \) is square-free, the resultant \( R(G, G') \) is an integer distinct from 0. It is a well-known fact that for every integer polynomial \( Q(x) \) with \( \deg Q(x) > 0 \) there are infinitely many primes \( p_1 < p_2 < p_3 < \cdots \) and
natural numbers $n_1, n_2, n_3, \ldots$ such that $p_i \mid Q(n_i)$ for $i = 1, 2, 3, \ldots$. Let $p_i$ and $n_i$ be such numbers in the case $Q = G$. Then,

$$G(n_i + p_i) - G(n_i) - p_iG'(n_i) = p_i^2(6a_4n_i^2 + 4a_4n_i + 3a_3 + a_2 + a_2p_i).$$

We deduce $p_i^2 \mid G(n_i + p_i) - G(n_i) - p_iG'(n_i)$; whence $p_i \mid G(n_i + p_i)$. Because $P(n_i + p_i)$ is a perfect square, we obtain $p_i^2 \mid G(n_i + p_i)$. Also, $p_i^2 \mid G(n_i)$. Hence, $p \mid G'(n_i)$. We conclude that, for $i = 1, 2, 3, \ldots$, the prime $p_i$ is a divisor of the integer $R(G, G')$. Consequently, $R(G, G') = 0$, a contradiction. This proves that $\deg G = 0$.

We obtain $P(x) = (F(x))^2$. Putting $F(x) = ux^2 + vx + w$, we have

$$ax^4 + bx^2 + c = (ux^2 + vx + w)^2.$$

Comparing coefficients finally yields (1).

**OC125.** $ABC$ is an acute angle triangle with $\angle A > 60^\circ$ and $H$ is its orthocenter. $M, N$ are two points on $AB, AC$ respectively, such that $\angle HMB = \angle HNC = 60^\circ$. Let $O$ be the circumcenter of triangle $HMN$. Let $D$ be a point on the same side of $BC$ as $A$ such that $\triangle DBC$ is an equilateral triangle. Prove that $H, O, D$ are collinear.

*Originally question 1 from 2012 Chinese Team Selection Test, day 1.*

*We give the solution of Oliver Geupel.*

Let $\rho$ be the rotation about the fixed point $B$ by an angle of $60^\circ$ such that $\rho(D) = C$. Let $\rho(H) = H'$.

We have

$$\frac{HM}{HN} = \frac{HB}{HC} = \frac{HH'}{HC} \quad \text{and} \quad \angle MHN = \angle BHC - 60^\circ = \angle H'HC.$$
Hence, the triangles $MHN$ and $H'HC$ are similar. Moreover, by rotation,

$$\angle HH'C = \angle BH'C - 60^\circ = \angle BHD - 60^\circ.$$ 

Thus,

$$\angle HNM = \angle HCH' = 180^\circ - \angle H'HC - \angle HH'C = 300^\circ - \angle BHC - \angle BHD = \angle CHD - 60^\circ.$$ 

We obtain

$$\angle MHO = \frac{1}{2}(180^\circ - \angle HOM) = 90^\circ - \angle HNM = 150^\circ - \angle CHD = \angle MHD.$$ 

Consequently, the points $H$, $O$, and $D$ are collinear.

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**Meetings at sea**

Yes, weekly from Southampton
Great steamers, white and gold,
   Go rolling down to Rio
And I’d like to roll to Rio.
   Some day before I’m old!

So, weekly, say every Thursday at noon, steamers leave Southampton to roll down to Rio. This trip of total length 9800 kilometres takes a steamer exactly 14 days to complete (so it covers the distance of 700 kilometres a day) and it arrives to Rio at noon on Thursday, two weeks later. After a 4-day stop in Rio, the steamer sails back and in exactly 14 days at noon on Monday it arrives to Southampton. In 3 more days (note - again on Thursday!), the steamer rolls to Rio. And I’d like to roll to Rio. So I board a steamer in Southampton and sail over to Rio. You task is to find out:

a) How many steamers rolling back to Southampton will I see on my trip?
b) When (which days of the week) will I see those steamers?
c) How far away from Southampton will each one of them be?
d) Two steamers meet at sea. Is it true that at the same time at some other place at sea two other steamers meet? If yes, then what is the distance between these two meeting points?
e) How many steamers sail back and forth between Southampton and Rio?

*From article by A. Rosenthal, Kvant, 1976 (5). Poem is by Rudyard Kipling.*