FOCUS ON...
No. 12
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Intersecting Circles and Spiral Similarity

Introduction
Let two circles $C_1, C_2$, with centres $O_1, O_2$ and radii $r_1, r_2$, intersect at points $U, V$. Among the spiral similarities transforming $C_1$ into $C_2$, those with centre $U$ or $V$ deserve a special interest. Specifically, let $\sigma$ be the one with centre $U$. Of course, the factor of $\sigma$ is $\frac{r_2}{r_1}$ and $\sigma(O_1) = O_2$, so that its angle is $\theta = \angle(UO_1, UO_2)$, the directed angle from vector $\overrightarrow{UO_1}$ to vector $\overrightarrow{UO_2}$. But a special feature of this transformation, emphasized in this number, is the very simple way the image of any point $P$ of $C_1$ is obtained: $P' = \sigma(P)$ is the second point of intersection of the circle $C_2$ with the line through $P$ and $V$.

The proof is easy. First note that $P'$ certainly is on $C_2$, hence we just have to show that $P, V, P'$ are collinear. With the help of the usual properties of angles subtending arcs of a circle, we calculate

$$\angle(VP, V') = \angle(VP, VU) + \angle(VU, V') = \frac{1}{2} \angle(O_1 \overrightarrow{P}, O_1 \overrightarrow{U}) + \frac{1}{2} \angle(O_2 \overrightarrow{U}, O_2 \overrightarrow{P}).$$

Since a spiral similarity preserves directed angles, we have

$$\angle(O_2 \overrightarrow{U}, O_2 \overrightarrow{P'}) = \angle(O_1 \overrightarrow{U}, O_1 \overrightarrow{P})$$
and finally $\angle(VP, V') = 0 \pmod{\pi}$. The conclusion follows.

In the proof above, it is understood that $P$ is different from $V$. But as $P$ approaches $V$ on $C_1$, the limiting position of the line $VP$ is the tangent to $C_1$ at $V$. Therefore $V' = \sigma(V)$ is the point where this tangent meets $C_2$ again (see figure below).
For convenience, this result about spiral similarities will be called \((R)\) in what follows.

**Two applications**

(a) Keeping the above notations, let \(\lambda\) be a fixed real number and suppose that an arbitrary line \(\ell\) through \(V\) meets again \(C_1\) at \(P_1\) and \(C_2\) at \(P_2\). What is the locus of \(R_\lambda = \lambda P_1 + (1 - \lambda) P_2\) as \(\ell\) turns around \(V\)? (This is, slightly modified, Walther Janous’s problem 2706 [2002 : 54; 2003 : 54]). Here is a simple solution based on the properties of spiral similarities and prompted by \((R)\). Let \(\ell_0 = P_0^0\) be the position of \(\ell\) parallel to \(O_1 O_2\) and let \(R_\lambda^0 = \lambda P_1^0 + (1 - \lambda) P_2^0\). Then, from \((R)\), \(\sigma(P_1^0) = P_2^0\) and for any other position of \(\ell\), \(\sigma(P_1) = P_2\). Thus, the spiral similarity with centre \(U\) transforming \(P_1^0\) into \(P_1\) also transforms \(P_0^0\) into \(P_2\) and, as it preserves collinearity and signed ratio, transforms \(R_\lambda^0\) into \(R_\lambda\). As a result, the spiral similarity \(\sigma_\lambda\) with centre \(U\) such that \(\sigma_\lambda(P_1^0) = R_\lambda^0\) satisfies \(\sigma_\lambda(P_1) = R_\lambda\). Since \(P_1\) traverses \(C_1\) as \(\ell\) varies, the locus of \(R_\lambda\) is \(\sigma_\lambda(C_1)\) that is, the circle with centre \(\sigma_\lambda(O_1)\) passing through \(U\) (note that \(R_\lambda = U\) when \(P_1 = U\)).

(b) As a second example where a call to \((R)\) is quite natural, consider the following question (extracted from [1]) :

Let \(A, B, C, D\) be four concyclic points such that \(AC, BD\) intersect at \(E\) and \(AD, BC\) intersect at \(F\). If \(C, D, E, F\) are concyclic, show that \(EF\) is perpendicular to \(AB\).

Consider the spiral similarity \(\sigma\) with centre \(C\) transforming the circle \((CDEF)\) into the circle \((ABCD)\). From \((R)\), we have \(\sigma(E) = B\) and \(\sigma(F) = A\) and it follows that \(\angle(CE, CB) = \angle(CF, CA)\). Hence \(\angle(CE, CB) = \angle(CB, CA)\) and, since \(E, C, A\) are collinear, the angle of \(\sigma\) must be a right angle. In consequence, \(EF\) is perpendicular to its image \(AB\).

![Diagram](image)

**Coming across Ptolemy in Croatia**

Our last example is #3 of the Croatian Mathematical Olympiad 2006 [2009 : 293] :

The circles \(\Gamma_1\) and \(\Gamma_2\) intersect at the points \(A\) and \(B\). The tangent line to \(\Gamma_2\) through the point \(A\) meets \(\Gamma_1\) again at \(C\) and the tangent line
to \( \Gamma_1 \) through \( A \) meets \( \Gamma_2 \) again at \( D \). A half-line through \( A \), interior to the angle \( \angle CAD \), meets \( \Gamma_1 \) at \( M \), meets \( \Gamma_2 \) at \( N \), and meets the circumcircle of \( \triangle ACD \) at \( P \). Prove that \( AM = NP \).

Amengual Covas’s neat solution is based on similar triangles [2010 : 444]; we propose a variant using property \( (R) \).

Let \( \Gamma \) be the circumcircle of \( \triangle ACD \) and \( O_1, O_2, O \) be the centres of \( \Gamma_1, \Gamma_2, \Gamma \), respectively. Let \( \sigma_B \) denote the spiral similarity with centre \( B \) transforming \( \Gamma_1 \) into \( \Gamma_2 \). From \( (R) \), we have \( \sigma_B(M) = N \) and \( \sigma_B(A) = D \), hence

\[
\frac{ND}{AM} = \frac{r_2}{r_1} \tag{1}
\]

where \( r_1, r_2 \) are the radii of \( \Gamma_1, \Gamma_2 \), respectively.

In the same way, if \( \sigma_D \) is the spiral similarity with centre \( D \) transforming \( \Gamma_2 \) into \( \Gamma \), we have \( \sigma_D(N) = P \) and \( \sigma_D(O_2) = O \). Thus,

\[
\frac{ND}{NP} = \frac{O_2D}{O_2O} = \frac{O_2D}{AO_1} = \frac{r_2}{r_1}
\]

(note that \( AO_2OO_1 \) is a parallelogram, its sides being parallel). A comparison with \( (1) \) gives \( AM = NP \), as desired.

And what about Ptolemy? Well, in some way, his famous theorem is hidden in this problem! Ptolemy’s Theorem states that if \( A, B, C, D \) are four points in this order on a circle, then \( AB \cdot CD + BC \cdot AD = AC \cdot BD \). The following proof is closely related to the above problem.

Let \( \Gamma \) be the circumcircle of \( ABCD \) and let \( \sigma_A \) be the spiral similarity with centre \( A \) such that \( \sigma_A(C) = D \). Let \( \sigma_A(B) = E \). If \( \Gamma_1 \) is the circumcircle of triangle \( ADE \), we have \( \Gamma_1 = \sigma_A(\Gamma) \) and property \( (R) \) tells us that \( E \) is on \( BD \) (between \( B \) and \( D \) as \( \angle CAD < \angle BAD \)). Similarly, if \( \sigma_C \) is the spiral similarity with centre \( C \) such that \( \sigma_C(A) = D \), then \( F = \sigma_C(B) \) is on \( \Gamma_2 = \sigma_C(\Gamma) \) and on the line segment \( BD \).

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Now, because of (R) again, \( CD \) (resp. \( AD \)) is tangent at \( D \) to \( \Gamma_1 \) (resp. \( \Gamma_2 \)), so we recognize the configuration of the problem and derive \( DE = FB \).

To conclude, since \( \frac{AD}{AC} = \frac{DE}{BC} \) and \( \frac{CD}{CA} = \frac{DF}{AB} \), we have
\[
AB \cdot CD + BC \cdot AD = AC(DF + DE) = AC(DF + FB) = AC \cdot BD.
\]

**Exercise**

The result (R) should help the reader to solve the following problem, adapted from Gerry Leversha’s problem 2457 [1999 : 308; 2000 : 316]). Let \( ABCD \) be a quadrilateral such that \( AD \) and \( BC \) intersect at \( E \). Suppose that \( ID = IC, JA = JC, KB = KD \) and \( \angle(\overrightarrow{ID}, \overrightarrow{IC}) = \angle(\overrightarrow{JA}, \overrightarrow{JC}) = \angle(\overrightarrow{KD}, \overrightarrow{KB}) = \angle(\overrightarrow{EA}, \overrightarrow{EB}) \). Show that \( E, I, J, K \) are collinear.

**Reference**