THE CONTEST CORNER

No. 25
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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by September 1, 2015, although late solutions will also be considered until a solution is published.

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CC121. Towns A and B are situated on two straight roads intersecting at the angle of \(\angle ACB = 60^\circ\). One way to get from A to B is by taking the bus which goes from A to C to B; this takes 11 minutes. Alternatively, you can walk from A directly to B, which takes an hour and 10 minutes. Finally, you can first walk from A to the road on which B is situated and then take the bus to B, but this takes longer still even if the bus comes immediately.

Find the distance from A to the intersection C if you walk at the speed of 3 km/h and the bus drives at the speed of 30 km/h.

CC122. The sequence \(\{x_n\}\) is given by the following recursion formula:

\[
x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2, \quad 0 < a < 1.
\]

Find the limit of the sequence.

CC123. Find how many pairs of integers \((x, y)\) satisfy the inequality

\[
2^{x^2} + 2^{y^2} < 2^{1976}.
\]

CC124. In a chess tournament, every participant played every other participant exactly once. In each game each participant scored 1 point for the win, 0.5 points for the tie and 0 points for the loss. At the end of the tournament, you discovered that in any group of any three participants there is one who, in the games against the other two, got 1.5 points. What is the maximum possible number of participants the tournament could have had?

CC125. Orthogonal projections of a triangle \(ABC\) onto two perpendicular planes are equilateral triangles with side length 1. If the median \(AD\) of triangle \(ABC\) has length \(\sqrt{\frac{a}{3}}\), find \(BC\).

Crux Mathematicorum, Vol. 40(5), May 2014
CC121. On considère deux villes $A$ et $B$. Chacune est située sur une route droite. Les deux routes se coupent en $C$ de manière que l’angle $ACB$ mesure $60^\circ$. Pour se rendre de $A$ à $B$, Dan peut prendre l’autobus qui va de $A$ à $C$ à $B$, ce qui prend 11 minutes. Dan peut aussi marcher directement de $A$ à $B$, ce qui prend une heure et 10 minutes. Enfin, Dan peut marcher de $A$ jusqu’à la route sur laquelle $B$ est située, puis prendre l’autobus jusqu’à $B$, mais cela prend encore plus de temps, même si l’autobus arrive immédiatement.

Déterminer la distance de $A$ à $C$, sachant que Dan marche à une vitesse de $3$ km/h et que l’autobus avance à une vitesse de $30$ km/h.

CC122. La suite $\{x_n\}$ est définie de façon récursive, $a$ étant un nombre réel, $0 < a < 1$:

$$x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2.$$ 

Déterminer la limite de la suite.

CC123. Combien y a-t-il de couples $(x, y)$ d’entiers qui vérifient l’équation $2x^2 + 2y^2 < 2^{1976}$?

CC124. Dans un tournoi d’échecs, chaque participant a rencontré chaque autre participant exactement une fois pour une partie. Après chaque partie, on a attribué 1 point au gagnant et 0 point au perdant et dans le cas d’un match nul, on a attribué 0,5 point à chaque participant. À la fin du tournoi, on constate que dans n’importe quel groupe de trois participants, il y a un participant qui a marqué un total de 1,5 point dans ses parties contre les deux autres. Quel est le nombre maximal possible de participants dans ce tournoi?

CC125. Chacune des projections orthogonales d’un triangle $ABC$ sur deux plans mutuellement orthogonaux est un triangle équilatéral avec des côtés de longueur 1. Sachant que la médiane $AD$ du triangle $ABC$ a une longueur de $\sqrt{\frac{3}{2}}$, déterminer la longueur du côté $BC$.  

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CONTEST CORNER
SOLUTIONS

CC71. A bag is filled with red and blue balls. Before drawing a ball, there is a \( \frac{1}{4} \) chance of drawing a blue ball. After drawing out a ball, there is now a \( \frac{1}{5} \) chance of drawing a blue ball. How many red balls are in the bag?

*Originally problem 10 from 2012 W.J. Blundon Mathematics Contest.*

*We present the solution of Petros Souldis.*

Let \( n \) be the total number of balls and \( x \) the number of blue balls. For picking the first ball, we have

\[
\frac{1}{4} = P(\text{first ball is blue}) = \frac{x}{n}.
\]

Thus, \( n = 4x \). For the second ball, we consider two cases based on whether the first selected ball is blue or red.

**Case 1:** The first ball picked was red. Then we have:

\[
\frac{1}{5} = P(\text{second ball is blue}) = \frac{x}{n-1}
\]

This gives \( 5x = n - 1 \) and substituting in \( x = 4n \), we get \( x = -1 \), so this cannot occur.

**Case 2:** The first ball picked was blue. Then we have:

\[
\frac{1}{5} = P(\text{second ball is blue}) = \frac{x - 1}{n - 1}
\]

This gives \( 5x - 5 = n - 1 \) and if we substitute in \( x = 4n \) we get \( x = 4 \). The total number of balls will be \( n = 4x = 16 \) and the number of red balls will be \( 16 - 4 = 12 \).

CC72. From the set of natural numbers \( 1, 2, 3, \ldots, n \), four consecutive even numbers are removed. The remaining numbers have an average value of \( 51 \frac{9}{16} \). Determine all sets of four consecutive even numbers whose removal creates this situation.

*Originally 1995 Invitational Mathematics Challenge, Grade 11, problem 4.*

*We present the solution by Titu Zeonaru and Neculai Stanciu.*

If \( n > 107 \), when removing the largest 4 integers, \( (n, n-1, n-2, n-3) \), the average of the remaining numbers is \( (n - 3)/2 > 52 \).

If \( n < 99 \), even when removing the smallest 4 integers, \( (1, 2, 3, 4) \), the average is at most 51.5.
Suppose the numbers we remove are \(2k, 2k + 2, 2k + 4, 2k + 6\). Then the total is
\[
\frac{n(n + 1)/2 - 8k - 12}{n - 4} = \frac{n^2 + n - 16k - 24}{2n - 8}.
\]
Since the numerator of this fraction is an integer when \(n\) and \(k\) are integers, for this to give an answer that, in reduced form, has a denominator of 16, we must have \(2n - 8 \equiv 0 \pmod{16}\), or \(n \equiv 4 \pmod{8}\). The only possible \(n\) of that form in the range \([99, 107]\) is \(n = 100\). When \(n = 100\), we get that \(k = 11\), so \(22, 24, 26, 28\) is the only possible set of removed numbers.

**CC73.** A farmer owns a triangular field, as shown. He reckons 5 sheep can graze in the west field, 10 sheep can graze in the south field, and 8 can graze in the east field. (All sheep eat the same amount of grass.) How many sheep can graze in the north field?

![Diagram of a triangular field with sheep grazing areas labeled W, S, E, and N.](image)

*Originally problem 9 from 2012 W.J. Blundon Mathematics Contest.*

*We present the solution by Šefket Arslanagić.*

We label the triangle as in the diagram:

![Labelled diagram of the triangular field](image)

We will now repeatedly use the result that if two triangles have the same height, the ratio of their areas is equal to the ratio of their bases:
\[
\frac{x_1 + 5}{x_2} = \frac{AF}{FE} = \frac{10}{8},
\]

\[
\frac{x_1}{5} = \frac{CD}{AD} = \frac{x_1 + x_2 + 8}{15}
\]

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Solving these two equations for \(x_1\) and \(x_2\) yields \(x_1 = 10, x_2 = 12\), so \(N\) can hold \(10 + 12 = 22\) sheep.

**CC74.** Let \(1000 \leq n = ABCD_{10} \leq 9999\) be a positive integer whose digits \(ABCD\) satisfy the divisibility condition:

\[
1111 \mid (ABCD + AB \times CD).
\]

Determine the smallest possible value of \(n\).

*Originally 2014 Sun Life Financial Répêchage Competition, problem 3.*

Solved by Richard Hess; David Manes; and Titu Zvonaru and Neculai Stanciu. Here is the summary of all the solutions.

The answer is 1729, and 1729 + 17 \(\times\) 29 = 2222. Other multiples of 1111 along with their corresponding numbers are (4444, 3142), (5555, 3845), (6666, 3399), (8888, 8307), (11110, 5890), (11110, 9418), (13332, 7289), (13332, 9146).

It is straightforward to see that 1111 cannot be represented in the form \(ABCD + AB \times CD\). Suppose that \(A = 1\) and \(0 \leq B \leq 6\). Then

\[
(ABCD)_{10} = (100)(10A + B) + (10A + B + 1)(10C + D) \\
= (100)(10 + B) + (11 + B)(10C + D)
\]

cannot exceed 1600 + 1700 = 3300 and so must be equal to 2222. Taking account of the fact that 2222 − 100(10 + B) must be divisible by 11 + B with a quotient not exceeding 99, we see that there are no possibilities for \(0 \leq B \leq 6\).

Now consider \(A = 1\) and \(B = 7\). Since 2222 − 1700 = 522 = 18 \(\times\) 29, we find that \(ABCD = 1729\) works. Since 3333 − 1700 is not a multiple of 18, this is the only possibility with \((A, B) = (1, 7)\).

**CC75.** Let \(P\) be a point inside the triangle \(ABC\) such that \(\angle PAC = 10^\circ\), \(\angle PCA = 20^\circ\), \(\angle PAB = 30^\circ\) and \(\angle ABC = 40^\circ\). Determine \(\angle BPC\).

*Originally 2004 MUN Undergrad Math Competition, Question 7.*

We present 4 solutions.

*Solution 1, by Miguel Amengual Covas.*

Since \(\angle CAB = \angle CBA = 40^\circ\), triangle \(ABC\) is isosceles and symmetric about its altitude from \(C\) to \(AB\). The reflection in this altitude fixes \(C\), switches \(A\) and \(B\), and carries \(P\) to a point \(Q\). In particular, \(\angle CQB = \angle CPA = 150^\circ\). Since \(CP = CQ\) and \(\angle PCQ = \angle ACB - 2\angle ACP = 100^\circ - 40^\circ = 60^\circ\), triangle \(CPQ\) is equilateral.

Therefore \(PQ = CQ\). Since also \(\angle PQB = 360^\circ - \angle PQC - \angle CQB = 360^\circ - 60^\circ - 150^\circ = 150^\circ = \angle CQB\), and \(QB\) is common, triangles \(PQB\) and \(CQB\) are congruent. Therefore \(BP = BC\) and so \(\angle BPC = \angle BPQ + \angle CPQ = \angle BCQ + 60^\circ = 80^\circ\).
Solution 2, by Michel Bataille, and David E. Manes (independently).

Let \( c = |AC| = |BC|, u = |CP|, \) and \( v = |BP| \). By the Sine Law applied to triangle \( APC \), \( u = c \sin 10^\circ / \sin 150^\circ = 2c \sin 10^\circ = 2c \cos 80^\circ \). By the Cosine Law applied to triangle \( BPC \), \( v^2 = c^2 + u^2 - 2uc \cos 80^\circ = c^2 \). Therefore \( v = c \), triangle \( BCP \) is isosceles and \( \angle BPC = \angle BCP = 80^\circ \).

Solution 3, by Šefket Arslanagić.

Since triangle \( ABC \) is isosceles, the altitude \( CN \) from \( C \) to \( AB \) bisects angle \( ACB \). Thus, \( \angle PCN = \angle ACN - \angle PCA = 50^\circ - 20^\circ = 30^\circ \). Let the line \( AP \) meet \( CN \) at \( M \). Since \( \angle CPM = \angle PCM = 30^\circ \), \( MP = CM \). Since \( \angle PMC = 120^\circ \), it follows that \( \angle PMN = 60^\circ \), so that \( \angle PMB = \angle CMB = 120^\circ \). Triangles \( PMB \) and \( CMB \) are congruent (SAS), so that \( PB = CB \) and \( \angle BPC = \angle BCP = 80^\circ \).

Solution 4, by George Apostolopoulos.

Let \( x = \angle PBC \). Using the Sine Law, we have that

\[
1 = \frac{PA}{PB} \cdot \frac{PB}{PC} \cdot \frac{PC}{PA} = \frac{\sin(40^\circ - x)}{\sin 30^\circ} \cdot \frac{\sin 80^\circ}{\sin 10^\circ} \cdot \frac{\sin 10^\circ}{\sin 20^\circ},
\]

from which \( \sin x \sin 20^\circ = 2 \sin(40^\circ - x) \cos 10^\circ \sin 10^\circ \). Thus \( \sin x = \sin(40^\circ - x) \) and \( x = 20^\circ \). Hence

\[
\angle BPC = 180^\circ - \angle PBC - \angle PCB = 180^\circ - 20^\circ - 80^\circ = 80^\circ.
\]