SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3821. Proposed by Farrukh Rakhimjanovich Ataev.

Prove that any triangle can be divided into five triangles such that one of the triangles is equilateral, one is isosceles, one is right angled, one is acute and one is obtuse.

Solved by P. Woo; and by T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Titu Zvonaru and Neculai Stanciu, slightly modified by the editor.

Let $\triangle ABC$ be the given triangle. We will use the following claim: “If $\triangle XYZ$ is a right nonisosceles triangle and $T$ is the midpoint of the hypotenuse $YZ$, then of the two isosceles triangles $\triangle XTZ$ and $\triangle XTY$, one is acute and the other is obtuse.”

The claim can be verified by considering a circle with diameter $YZ$ to see that $|TX| = |TY| = |TZ|$. Consider two cases:

(a) If $\triangle ABC$ is equilateral, let $M$ be the midpoint of $BC$, $N$ the midpoint of $AB$, $E$ the midpoint of $AM$, and $D$ the projection of $M$ onto $AC$. We have $\triangle BMN$ equilateral, $\triangle ANM$ obtuse, $\triangle MDC$ right-angled. By the claim, one of $\triangle MDE$ and $\triangle ADE$ is acute and the other is isosceles. (One can check that $\triangle MDE$ is the acute triangle, as it is in fact equilateral.)

(b) If $\triangle ABC$ is not equilateral, let $A$ be the largest angle, so $A > 60^\circ$. Without loss of generality, suppose $B < 60^\circ$. Let $M$ be a point on $BC$ such that $\angle BAM = 60^\circ$. Since $B < 60^\circ$, $AB > AM$, and we can find a point $N$ on $AB$ for which $|AN| = |AM|$. Then $\triangle ANM$ is equilateral and $\triangle BNM$ is obtuse. Let $D$ be the projection of $M$ onto $AC$, which lies on $AC$ because $C < 90^\circ$. If at least one of the two right triangles $\triangle MDC$ and $\triangle ADM$ is nonisosceles, then by the claim, we can divide it into an acute triangle and an isosceles triangle. If both $\triangle MDC$ and $\triangle ADM$ are isosceles, then $C = 45^\circ$, $A = 60^\circ + 45^\circ$, and $B = 30^\circ$, in which case we swap the roles of $B$ and $C$ in this argument.

3822. Proposed by M. N. Deshpande.

Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle A = \alpha$. Further, let $G$ be its centroid and circle $\Gamma$ passes through $B$, $C$ and $G$. Point $D$ is on the circle, different from $G$, such that $BD = CD$ and let $\angle BDC = \delta$. Show that

(i) $\alpha + \delta \geq 120^\circ$.

(ii) $\left( \frac{\cos \alpha + \cos \delta}{1 + \cos \alpha \cos \delta} \right)$ does not depend on $\alpha$. 

Copyright © Canadian Mathematical Society, 2015
(i) Let $\angle GCA = \psi$. Since $BDCG$ is cyclic, the angles at $B$ and $C$ sum to $180^\circ$, so that (in quadrilateral $BDCA$) $\alpha + \delta + 180^\circ + 2\psi = 360^\circ$. To prove $\alpha + \delta \geq 120^\circ$, therefore, we need only to prove that $\psi \leq 30^\circ$.

Let $GT$ be the perpendicular from $G$ to $AC$. Then $GT = \frac{1}{3}m_b = \frac{1}{3}m_c$, whence

$$\sin \psi = \frac{GT}{GC} = \frac{GT}{\frac{1}{3}m_c} \leq \frac{\frac{1}{3}m_c}{\frac{1}{3}m_c} = \frac{1}{2}.$$ 

Hence $\sin \psi \leq \frac{1}{2} = \sin 30^\circ$. Since the function $y = \sin x$ is increasing on the interval $(0, \frac{\pi}{2})$ we get $\psi \leq 30^\circ$, as desired.

(ii) Observe that $\angle GBC = \angle GCB = \frac{1}{2} \angle BDC = \frac{1}{2} \delta$.

Thus

$$\tan \frac{\delta}{2} = \frac{\frac{1}{3}m_a}{\frac{2}{3}a} = \frac{2m_a}{3a} \quad \text{and} \quad \tan \frac{\alpha}{2} = \frac{a}{m_a} = \frac{a}{2m_a}.$$ 

Therefore

$$\cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} = \frac{4m_a^2 - a^2}{4m_a^2 + a^2} \quad \text{and} \quad \cos \delta = \frac{1 - \tan^2 \frac{\delta}{2}}{1 + \tan^2 \frac{\delta}{2}} = \frac{9a^2 - 4m_a^2}{9a^2 + 4m_a^2},$$ 

and, finally,

$$\frac{\cos \alpha + \cos \delta}{1 + \cos \alpha \cos \delta} = \frac{64a^2m_a^2}{80a^2m_a^2} = \frac{4}{5},$$ 

which, indeed, does not depend on $\alpha$.

3823. Proposed by Neculai Stanciu and Titu Zvonaru.

Let $ABC$ be a triangle with height $AD$, where $E$ and $F$ are the midpoints of sides $AC$ and $AB$ respectively. For any point $P$ in the plane of the triangle $ABC$, let $Y$ and $Z$ be its symmetric from the points $E$ and $F$, respectively. If $P'$ is the midpoint of $DP$ and $M = BY \cap CZ$, then prove that the line through $M$ and $P'$ passes through a fixed point.

Solved by M. Bataille; O. Geupel; O. Kouba; M.R. Modak; C. Sánchez-Rubio; P. Woo; and the proposers. We present the solution by Omran Kouba.
There is no need to restrict $D$ to be the foot of the altitude from $A$: We shall see that for any point $D$ in the plane of the triangle $ABC$, the barycenter of the points $A, B, C,$ and $D$ lies on all the lines $MP'$.

Since $\overrightarrow{PY} = 2\overrightarrow{PE}$ and $\overrightarrow{PZ} = 2\overrightarrow{PF}$, we conclude that

$$\overrightarrow{YZ} = 2(\overrightarrow{PF} - \overrightarrow{PE}) = 2\overrightarrow{EF} = \overrightarrow{CB}.$$  

Thus $BCYZ$ is a parallelogram, and $M$ is the intersection point of its diagonals, so $M$ is the midpoint of $BY$. Therefore

$$2(\overrightarrow{DP'} + \overrightarrow{DM}) = \overrightarrow{DP} + \overrightarrow{DY} + \overrightarrow{DB} \quad (P' \text{ and } M \text{ are the midpoints of } DP \text{ and } BY)$$

$$= 2\overrightarrow{DE} + \overrightarrow{DB} \quad (E \text{ is the midpoint of } PY)$$

$$= \overrightarrow{DA} + \overrightarrow{DC} + \overrightarrow{DB} \quad (E \text{ is the midpoint of } AC).$$

Thus, if $X$ is the midpoint of $P'M$ then we conclude from the previous result that

$$4\overrightarrow{DX} = \overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC} + \overrightarrow{DD}.$$  

This means that $X$ is the barycenter of the four points $A, B, C,$ and $D$, and all the lines through $M$ and $P'$ pass through this fixed point (which is the common midpoint of all segments $MP'$).

Editor’s comment. Bataille observed that $M$ is not well defined should $P$ be chosen on the line through $A$ that is parallel to $BC$. In that case the result continues to hold if $M$ is defined to be the common midpoint of $BY$ and $CZ$.


Let

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

where $n \in \mathbb{N}$. It is well known that $S_n \geq 2(\sqrt{n} + 1 - 1)$. Prove or disprove the stronger inequality that

$$S_n \geq \frac{2n}{1 + \sqrt{n}}.$$  

Solved by AN-anduud Problem Solving Group; S. Arslanagić; M. Bataille; M. Dineč; O. Koubá; K-W. Lau; S. Malikić; C.R. Pranesachar; N. Stanciu and T. Zvonaru; D. Vâncu; P. Y. Woo; and the proposers. There was one incorrect submission. We present two solutions.

Solution 1, by Omran Kouba.

We will prove that the stronger inequality is valid for every positive integer $n$. Indeed, note that for positive $a$ and $b$ we have

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 4 + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \geq 4,$$
or equivalently, we obtain
\[ \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) \geq 2 \left( \frac{a - b}{a^2 - b^2} \right). \]

Taking \( a = \sqrt{k+1} \) and \( b = \sqrt{k} \), we find that for \( k \geq 1 \) we have
\[ \frac{1}{2} \left( \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \right) \geq 2 \left( \sqrt{k+1} - \sqrt{k} \right). \]

Adding these inequalities for \( k = 1, 2, \ldots, n-1 \) we get
\[ S_n - \frac{1}{2} \left( 1 + \frac{1}{\sqrt{n}} \right) \geq 2(\sqrt{n} - 1) = \frac{2(n-1)}{\sqrt{n} + 1}, \]

or equivalently
\[ S_n \geq \frac{2n}{\sqrt{n} + 1} + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{n}} - \frac{4}{\sqrt{n} + 1} \right) = \frac{2n}{\sqrt{n} + 1} + \frac{(\sqrt{n}-1)^2}{2(n+\sqrt{n})}. \]

This proves the desired inequality, with equality if and only if \( n = 1 \).

**Solution 2, by AN-anduud Problem Solving Group.**

Since the function \( f(x) = \frac{1}{\sqrt{x}} \) is strictly decreasing and strictly convex on \( x \geq 1 \),
inspection of a graph of the integral of \( \frac{1}{\sqrt{x}} \) and a left-endpoint Riemann sum for \( \frac{1}{\sqrt{x}} \) yields:

\[ S_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \int_1^n \frac{dx}{\sqrt{x}} + \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) + \frac{1}{\sqrt{n}} \]
\[ = 2(\sqrt{n} - 1) + \frac{1}{2} \left( 1 - \frac{1}{\sqrt{n}} \right) + \frac{1}{\sqrt{n}} \]
\[ = \frac{4n + 1 - 3\sqrt{n}}{2\sqrt{n}}. \]

Now we prove the following inequality for any positive integer \( n \),
\[ \frac{4n + 1 - 3\sqrt{n}}{2\sqrt{n}} \geq \frac{2n}{1 + \sqrt{n}}. \]

We have
\[ \frac{4n + 1 - 3\sqrt{n}}{2\sqrt{n}} \geq \frac{2n}{1 + \sqrt{n}} \Leftrightarrow (4n + 1 - 3\sqrt{n})(1 + \sqrt{n}) \geq 4n\sqrt{n} \]
\[ \Leftrightarrow (4n + 1) - 3\sqrt{n} + (4n\sqrt{n} + \sqrt{n} - 3n) \geq 4n\sqrt{n} \]
\[ \Leftrightarrow n + 1 - 2\sqrt{n} \geq 0 \]
\[ \Leftrightarrow (\sqrt{n} - 1)^2 \geq 0, \]

and this last inequality is clearly true. Hence by combining the two inequalities, we have \( S_n \geq \frac{2n}{1 + \sqrt{n}} \) for any positive integer \( n \).

Editor’s comments. The featured solutions were the only two submitted solutions that do not rely on induction. The induction proof is reasonably straightforward, although it is possible to reduce work by applying calculus techniques (as Bataille did using second derivatives and the Mean Value Theorem), or using rough approximations to eliminate the numerous square roots that appear (as Malikić did, bounding \( n(n + 1) \) and \( n + 1 \) above by squares).

3825. Proposed by Brian Brzycki.

Triangle \( ABC \) is acute. Points \( X \) and \( Y \) trisect side \( BC \), with \( X \) closer to \( B \). Semicircles centred at \( X \) and \( Y \) and tangent to \( AB \) and \( AC \) are drawn, respectively.

(a) Prove that the two semicircles must intersect.

(b) If the semicircles intersect at \( Z \), and \( \angle XZY = \theta \), prove that

\[
\cos(2B) + \cos(2C) + 4 \sin(B) \sin(C) \cos(\theta) = 0.
\]

Solved by M. Amengual Covas; G. Apostopoulos; Š. Arslanagić; M. Bataille; J. Heuver; O. Kouba; S. Malikić; C. R. Pranesachar; D. Smith; D. Stone and J. Hawkins; T. Zvonaru and N. Stanciu; P. Y. Woo; and the proposer. We present the solution given by most of the solvers.

(a) Without loss of generality, let \( BX = XY = YC = 1 \); let \( x \) and \( y \) be the respective radii of the semi-circles based on \( BC \) with centres \( X \) and \( Y \). Since the triangle is acute, \( B + C > 90^\circ \). Therefore,

\[
x + y = \sin B + \sin C = 2 \sin \frac{B + C}{2} \cos \frac{B - C}{2} > 2 \sin 45^\circ \cos 45^\circ = 1 = XY
\]

and the two semi-circles intersect.

(b) By the Law of Cosines applied to triangle \( ZXY \),

\[
1 = x^2 + y^2 - 2xy \cos \theta = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos \theta.
\]

Since \( \cos 2B = 1 - 2 \sin^2 B \) and \( \cos 2C = 1 - 2 \sin^2 C \), the desired result follows.

Editor’s comment. All the solvers used the Law of Cosines for part (b). However, for (a), Heuver, Kouba and the proposer used similar triangles to obtain

\[
x : XY = x : BX = h_c : BC,
\]

\[
y : XY = y : YC = h_b : BC,
\]

where \( h_c \) and \( h_b \) are the respective altitudes in triangle \( ABC \) from \( C \) and \( B \). Since these altitudes meet in the interior of the triangle, \( h_c + h_b > BC \), so that \( x + y > XY \).
 Proposed by Ovidiu Furdui.

Let \( f : [0, 1] \to [0, \infty) \) and let \( g : [0, 1] \to [0, \infty) \) be two continuous functions. Find the value of

\[
\lim_{n \to \infty} \sqrt{n} \left( f \left( \frac{1}{n} \right) g \left( \frac{2}{n} \right) + f \left( \frac{2}{n} \right) g \left( \frac{n-1}{n} \right) + \cdots + f \left( \frac{n}{n} \right) g \left( \frac{1}{n} \right) \right).
\]

Solved by O. Kouba; P. Perfetti; and the proposer. There was one incorrect submission. We present the solution by Omran Kouba, modified slightly by the editor.

Let \( \ell = \int_0^1 f(x)g(1-x)\,dx \). We will prove that if \( \ell > 0 \) then the considered limit is equal to 1, while the limit might not exist if \( \ell = 0 \). Indeed, let

\[
R_n = \sum_{k=1}^{n} f \left( \frac{k}{n} \right) g \left( 1 - \frac{k}{n} \right) \quad \text{and} \quad T_n = \sum_{k=1}^{n} f \left( \frac{k}{n} \right) g \left( 1 - \frac{k-1}{n} \right).
\]

By the Triangle Inequality, we have

\[
|R_n - T_n| \leq n \|f\|_\infty \omega \left( g, \frac{1}{n} \right),
\]

where \( \|f\|_\infty = \sup_{[0,1]} |f| \) and \( \omega(g, \epsilon) \) is a modulus of uniform continuity, given by

\[
\omega(g, \epsilon) = \sup \{|g(u) - g(v)| : (u, v) \in [0, 1]^2, |u - v| \leq \epsilon \}.
\]

(\( \omega(g, \epsilon) \) is finite because a continuous function on a closed and bounded interval is uniformly continuous.) Thus, since \( \lim_{\epsilon \to 0} \omega(g, \epsilon) = 0 \), we have

\[
\lim_{n \to \infty} \frac{|R_n - T_n|}{n} = 0.
\]

On the other hand, observe that \( \frac{1}{n} R_n \) is a Riemann sum for the function \( x \mapsto f(x)g(1-x) \), and consequently

\[
\lim_{n \to \infty} \frac{R_n}{n} = \int_0^1 f(x)g(1-x)\,dx = \ell.
\]

From these two equations, we conclude that

\[
\lim_{n \to \infty} \frac{T_n}{n} = \ell.
\]

Now, if \( \ell > 0 \), then there is \( n_0 \) such that for \( n \geq n_0 \) we have \( an \leq T_n \leq bn \), with \( a = \ell/2 \) and \( b = 3\ell/2 \). Thus

\[
\sqrt{an} \leq \sqrt{T_n} \leq \sqrt{bn}, \quad \text{for } n \geq n_0.
\]

But $\lim_{n \to \infty} \sqrt{cn} = 1$ for every $c > 0$, so by taking the limit as $n$ tends to $\infty$, we conclude that $\lim_{n \to \infty} \sqrt{T_n} = 1$, in the case $\ell > 0$.

Next, we consider the case $\ell = 0$, which is equivalent to $f(x)g(1-x) = 0$ for every $x \in [0,1]$, because the integrand is continuous and non-negative. We will give an example where the proposed limit does not exist. Indeed, let

$$f(x) = g(x) = \max \left( 0, x - \frac{1}{2} \right).$$

With this choice, note that $g(1-x) = \max(0, \frac{1}{2} - x)$, so that almost all of the terms in $T_n$ are zero. Treating even and odd $n$ separately, we have $T_{2n} = 0$ and

$$T_{2n+1} = f^2 \left( \frac{n + 1}{2n+1} \right) = \left( \frac{(2n + 2) - (2n + 1)}{2(2n+1)} \right)^2 = \frac{1}{4(2n+1)^2},$$

and therefore $\lim_{n \to \infty} 2\sqrt{T_{2n}} = 0$, and $\lim_{n \to \infty} 2^{n+1}/\sqrt{T_{2n+1}} = 1$, so, the proposed limit does not exist in this case.

Editor’s comments. This solution exhibits a reasonably important problem solving technique. If a quantity almost looks like it converges to something nice, but it doesn’t immediately do so, perhaps it’s a good idea to approximate it with something that converges in a more obvious fashion, and work from there. For example, here we are dealing with $T_n$, but $\frac{1}{n}T_n$ isn’t exactly a Riemann sum for that integral. However, $\frac{1}{n}R_n$ is, and the two sums approximate each other.

3827. Proposed by Jung In Lee.

For integer $k$, let $f(k)$ be the largest prime factor of $k$. The sequences $\{a_n\}$, $\{b_n\}$ are defined by $a_0 = b_0 = pq$, $a_{n+1} = a_n + pf(a_n)$, $b_{n+1} = b_n + qf(b_n)$ for $n \geq 1$, for given positive integers $p$ and $q$. Prove that there are infinitely many pairs of integers $(c, d)$ that satisfy

$$\frac{a_c}{p} = \frac{b_d}{q}.$$

No solutions to this problem were received. The problem remains open.

3828. Proposed by George Apostolopoulos.

Let $ABC$ be an acute angled triangle with $\angle B = 2\angle C$ and altitude $AD$. Drop perpendiculars $DK$ and $DL$ from $D$ to the sides $AB$ and $AC$ respectively.

(a) Prove that $\sin A > \frac{2\sin^2 C}{1 + \cos C}$.

(b) If $\frac{AD}{KL} = \sqrt{5} - 1$, find the angles of the triangle $ABC$.

Solved by A. Alt; Š. Arslanagić; R. Barbara; M. Bataille; P. De; O. Geupel; O. Kouba; V. Konečný (2 solutions); K.-W. Lau; S. Malikić; M. R. Modak; C.R.
Pranesachar; D. Smith; D. Stone and J. Hawkins; G. Tsapakidis; H. Wang; T. Zvonaru; and the proposer. We present the solution by Kee-Wai Lau expanded by the editor.

Let \( \angle C = \theta \), so that \( \angle B = 2\theta \) and \( \angle A = \pi - 3\theta \).

a) Since \( \triangle ABC \) is an acute triangle, we have that

\[
2\theta < \frac{\pi}{2} \quad \text{and} \quad \pi - 3\theta < \frac{\pi}{2},
\]

which imply

\[
\frac{\pi}{6} < \theta < \frac{\pi}{4},
\]

so \( \sin \theta < \frac{1}{\sqrt{2}} \). Hence

\[
4\sin^2 \theta < 2 \quad \text{or} \quad 3 - 4\sin^2 \theta > 1,
\]

from which we have:

\[
\sin A = \sin 3\theta = 3\sin \theta - 4\sin^3 \theta = (3 - 4\sin^2 \theta)\sin \theta > \sin \theta
\]

\[
= \left(\frac{\sqrt{2}}{2} + 1\right)\sin \theta > \frac{2\sin \theta}{\sqrt{2} + 1} = \frac{2\left(\frac{1}{\sqrt{2}}\right)\sin \theta}{1 + \frac{1}{\sqrt{2}}} > \frac{2\sin^2 \theta}{1 + \cos \theta} = \frac{2\sin^2 C}{1 + \cos C},
\]

since \( \sin \theta < \frac{1}{\sqrt{2}} < \cos \theta \).

b) Since \( \triangle ACD \sim \triangle ADL \) and \( \triangle ABD \sim \triangle ADK \), we have \( \angle ADL = \angle ACD = \theta \) and \( \angle ADK = \angle ABD = 2\theta \). Since \( \angle AKD = \angle ALD = \frac{\pi}{2}, \) \( AKDL \) is a cyclic quadrilateral. Hence,

\[
\angle DKL = \angle DAL = \frac{\pi}{2} - \angle ADL = \frac{\pi}{2} - \theta.
\]

Applying the sine law to \( \triangle DKL \), we have

\[
AD = \frac{DL}{\cos \theta} = \frac{DL}{\cos \left(\frac{\pi}{2} - \angle DKL\right)} = \frac{DL}{\sin \angle DKL} = \frac{KL}{\sin \angle KDL} = \frac{KL}{\sin \angle A} = \frac{KL}{\sin 3\theta}.
\]

Crux Mathematicorum, Vol. 40(3), March 2014
So \[ \sin 3\theta = \frac{KL}{AD} = \frac{1}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{4}. \]

It is well-known that \( \sin 54^\circ = \frac{\sqrt{5} + 1}{4} \) (Editor’s comment: this can be derived by elementary procedure and several solvers provided a detailed proof for this.) Since \( \angle A < \frac{\pi}{2} \) and \( \sin \angle A = \sin (\pi - 3\theta) = \sin 3\theta = \frac{\sqrt{5} + 1}{4} \), we conclude that \( \angle A = 54^\circ = \frac{3\pi}{10} \), from which it follows that

\[ \angle C = \theta = \frac{1}{3}(\pi - \angle A) = \frac{1}{3} \left( \pi - \frac{3\pi}{10} \right) = \frac{7\pi}{30} = 42^\circ \]
and \( \angle B = 2\theta = \frac{7\pi}{15} = 84^\circ \).

3829. Proposed by Michel Bataille.

Let \( a, b, c \) be positive real numbers and \( \Delta = a^2 + b^2 + c^2 - (ab + bc + ca) \). Improve the well known inequality \( \Delta \geq 0 \) by proving that

\[ \Delta \geq \left( \frac{a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2}{a + b + c} \right)^{\frac{1}{2}}. \]

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; R. Barbara; D. Bailey, E. Campbell and C. Diminnie; M. Dincă; N. Evgenidis; O. Kouba; D. Koukakis; K. -W. Lau; S. Malikic; P. McCartney; C.R. Praneschar; D. Smith; T. Zvonaru and N. Stanciu; and the proposer. There was one flawed solution. The more efficient approaches are summarized below.

Preliminaries. We establish notation and basic facts. The summation sign will refer to cyclic sums:

\[ \sum f(a, b, c) = f(a, b, c) + f(b, c, a) + f(c, a, b). \]

\[ \Delta = a^2 + b^2 + c^2 - ab - bc - ca \]
\[ = (a - b)(a - c) + (b - c)(b - a) + (c - a)(c - b) \]
\[ = (a - b)(a - c) + (b - c)^2 = (b - c)(b - a) + (c - a)^2 = (c - a)(c - b) + (a - b)^2 \]
\[ = \frac{1}{2} \left[ (a - b)^2 + (b - c)^2 + (c - a)^2 \right] \geq 0. \]
Then we have:

\[ A = a(a - b)(a - c) + b(b - a)(b - c) + c(c - a)(c - b) \]
\[ = \sum a^3 - \sum (a^2b + ab^2) + 3abc. \]

\[ B = a(a - b)^2(a - c)^2 + b(b - a)^2(b - c)^2 + c(c - a)^2(c - b)^2 \]
\[ = \sum a^5 + \sum (a^3b^2 + a^2b^3) + 4 \sum a^3bc - 3 \sum ab^2c^2 - 2 \sum (a^4b + ab^4) \]
\[ = \Delta A. \]

Finally,
\[ \Gamma = (a + b + c)\Delta^2 - B = \Delta[\Delta(a + b + c) - A] = \Delta[\sum (a^2b + ab^2) - 6abc] \]
\[ = \Delta[(a + b)(b + c)(c + a) - 8abc]. \]

The problem requires it to be shown that \( \Gamma \geq 0 \). Equality will occur if and only if \( a = b = c \).

**Solution 1**, by Š. Arslanagić; Kee-Wai Lau; Salem Malikić; and Phil McCartney (all independently).

\[ \Gamma = \Delta(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - 6abc) \geq 0, \]
by the arithmetic-geometric means inequality.

**Editor’s comment.** E. Nikolaos used the fact that \( \Gamma = \Delta[(a + b)(b + c)(c + a) - 8abc] \) and noted that \( a + b \geq 2\sqrt{ab}, \ b + c \geq 2\sqrt{bc}, \ c + a \geq 2\sqrt{ca} \).

**Solution 2**, by Titu Zvonaru and Neculai Stanciu.

\[ \Gamma = a[\Delta - (a^2 - ab - ac + bc)][\Delta + (a^2 - ab - ac + bc)] \]
\[ + b[\Delta - (b^2 - bc - ba + ca)][\Delta + (b^2 - bc - ba + ca)] \]
\[ + c[\Delta - (c^2 - ca - cb + ab)][\Delta + (c^2 - ca - cb + ab)] \]
\[ = (ab^4 + a^4b + bc^4 + b^4c + ca^4 + c^4a) + 6(ab^2c^2 + a^2bc^2 + a^2b^3c) \]
\[ - 8(a^3bc + ab^3c + abc^3) \]
\[ = a(b - c)^4 + b(c - a)^4 + c(a - b)^4 \geq 0. \]

**Solution 3**, by the AN-anduud Problem Solving Group; and Dimitrios Koukakis (independently).

Observe that
\[ (uv + vw + wu)^2 = u^2v^2 + v^2w^2 + w^2u^2 \]
when \( u + v + w = 0 \). Therefore, setting \( (u, v, w) = (a - b, b - c, c - a) \), we obtain
\[ (a + b + c)\Delta^2 = (a + b + c)[(a - b)^2(a - c)^2 + (b - c)^2(b - a)^2 + (c - a)^2(c - b)^2] \]
\[ \geq a(a - b)^2(a - c)^2 + b(b - c)^2(b - a)^2 + c(c - a)^2(c - b)^2 = B. \]
Solution 4, by Omran Kouba, modified by the editor.
Without loss of generality, assume that \( a \geq b \geq c \). Then

\[
\Delta \geq (a-b)(a-c) \geq |b-a|(b-c);
\]
\[
\Delta \geq (c-a)(c-b) = (a-c)(b-c).
\]

Then \( \Delta^2 \) is not less than each of \((a-b)^2(a-c)^2\), \((b-a)^2(c-a)^2\) and \((c-a)^2(c-b)^2\).
Therefore \( \Delta^2 \) is not less than the weighted average \( B/(a+b+c) \) of these terms.

Solution 5, by Dionne Bailey, Elsie Campbell and Charles Dininnie.
Since \( 2\Delta = (a-b)^2 + (b-c)^2 + (c-a)^2 \), then
\[
4\Delta^2 = 4(a-b)^2(c-a)^2 + [(a-b)^2-(c-a)^2]^2 + (b-c)^4 + 2(a-b)^2(b-c)^2 + 2(b-c)^2(c-a)^2,
\]
so that \( \Delta^2 \geq (a-b)^2(a-c)^2 \).

Similarly, \( \Delta^2 \geq (b-c)^2(b-a)^2 \) and \( \Delta^2 \geq (c-a)^2(c-b)^2 \). Hence the right side of
the inequality does not exceed \((a+b+c)^{-1/2}(a\Delta^2 + b\Delta^2 + c\Delta^2)^{1/2} = \Delta \).

Solution 6, by Arkady Alt.
Note that
\[
B = \sum a(a-b)(a-c)[\Delta - (b-c)^2]
\]
\[
= \Delta \sum a(a-b)(a-c) + (a-b)(b-c)(c-a) \sum a(b-c)
\]
\[
= \Delta \sum a[\Delta - (b-c)^2] + 0 = \Delta^2(a+b+c) - \Delta \sum a(b-c)^2.
\]
Hence
\[
\Gamma = \Delta \sum a(b-c)^2 \geq 0.
\]

Solution 7, by C.R. Pranesachar.
\[
\Gamma = (a+b+c)\Delta^2 - B = \Delta[(a+b+c)B - A]
\]
\[
= (b+c)(a-b)^2(a-c)^2 + (a+c)(b-a)^2(b-c)^2 + (a+b)(c-a)^2(c-b)^2 \geq 0.
\]

3830. Proposed by Tigran Hakobyan.
Let \( a > 0 \). Define the sequence \( \{a_n\}_{n=0}^\infty \) of real numbers by
\[
a_1 = a, a_{n+1} = a_n + \{a_n\}, n \geq 1
\]
where \( \{x\} \) is the fractional part of \( x \). Find all \( a > 0 \) such that the sequence \( \{a_n\}_{n=0}^\infty \)
defined above is bounded.

Solved by A. Alt; R. Barbara; O. Kouba; K. Lewis; P. Perfetti; D. Stone and J. Haukins; D. Váčaru; and the proposer. Two incorrect solutions were received.
We present a composite of solutions by the listed solvers.

Copyright © Canadian Mathematical Society, 2015
We prove first that the sequence is bounded if and only if it is eventually an integer.

Suppose first that the sequence \( \{a_n\}_{n=0}^{\infty} \) is bounded. Since the sequence is monotone increasing, it is convergent and thus Cauchy. Thus, there is a positive integer \( N \) such that for all \( n \geq N \), \( |a_n - a_N| < \frac{1}{2} \) and \( a_n < a_N + 1 \). The first of these inequalities can be written in the equivalent form \( \{a_n\} < \frac{1}{2} \) for all \( n \geq N \). We have \( a_{N+1} = \lfloor a_{N} \rfloor + 2 \{a_{N}\} \), \( a_{N+2} = \lfloor a_{N} \rfloor + 4 \{a_{N}\} \), and

\[
\lfloor a_{N} \rfloor + 2^M \cdot \{a_{N}\} = a_{N+M} < a_N + 1 < \lfloor a_{N} \rfloor + 2
\]

for all positive integers \( M \). Hence,

\[
2^M \cdot \{a_{N}\} < 2
\]

for all \( M \), so \( \{a_{N}\} = 0 \). Thus \( a_N \) is an integer.

Conversely, suppose \( a_N \) is an integer, for some positive integer \( N \). Then \( a_n = a_N \) for all \( n \geq N \), so that the sequence is bounded above by \( a_N \).

Now, we show that the sequence is eventually an integer if and only if it starts with a dyadic rational.

1. Suppose \( a_d \) is an integer for some nonnegative integer \( d \). If \( d = 0 \), then \( a \) is an integer, and thus a dyadic rational. If \( d > 0 \), then \( a = a_0 = k + \frac{m}{2^d} \), where \( k \) and \( m \) are nonnegative integers.

2. Suppose that \( a \) is a dyadic rational. Write \( a = k + \frac{m}{2^d} \), where \( k \), \( m \), and \( d \) are nonnegative integers, with \( d \geq 1 \) and \( m = 0 \) or odd. Then there are odd integers \( m_1, \ldots, m_d \) such that \( \{a_1\} = \frac{m_1}{2^{d-1}} \), and for \( n < d \), \( \{a_n\} = \frac{m_n}{2^{n\times d}} \), so that \( \{a_{d-1}\} = \frac{1}{2} \) and \( \{a_d\} = 0 \). Hence \( a_d \) is an integer.