The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to crux-olympiad@cms.math.ca or mail them to the address inside the back cover. Electronic submissions are preferable.

Submissions of solutions. Each solution should be contained in a separate file named using the convention LastName_FirstName_OCProblemNumber (example Doe_Jane_OC1234.tex). It is preferred that readers submit a $\LaTeX$ file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.

To facilitate their consideration, solutions should be received by the editor by 1 July 2015, although late solutions will also be considered until a solution is published.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions’ section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, de l’Université Saint-Boniface à Winnipeg, for translations of the problems.

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**OC171.** Find all 3-digit numbers for which the ratio between the number and the sum of its digits is minimal.

**OC172.** Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

**OC173.** Each integer is coloured with one of two colours, red or blue. It is known that, for every finite set $A$ of consecutive integers, the absolute value of the difference between the number of red and blue integers in the set $A$ is at most 1000. Prove that there exists a set of 2000 consecutive integers in which there are exactly 1000 red numbers and 1000 blue numbers.

**OC174.** Suppose that $a$ and $b$ are two distinct positive real numbers with the property that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for all positive integers $n$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. Prove that $a$ and $b$ are integers.

*Crux Mathematicorum, Vol. 40(3), March 2014*
OC175. Suppose $O$ is the center of the circumcircle of triangle $ABC$. Let $P$ be the midpoint of the arc $BAC$ and $QP$ be a diameter. Let $I$ be the incentre of the triangle $ABC$ and let $D$ be the intersection of $PI$ and $BC$. The circumcircle of $\triangle AID$ and the extension of $PA$ meet at $F$. Let $E$ be a point on the line segment $PD$ such that $DE = DQ$. Let $r, R$ be the radius of the inscribed circle and circumcircle of $\triangle ABC$, respectively. If $\angle AEF = \angle APE$, prove that
\[
\sin^2(\angle BAC) = \frac{2r}{R}.
\]

OC171. Déterminez tous les nombres à 3 décimales tels que le ratio entre le nombre et la somme de ses chiffres est minimal.

OC172. Déterminer tous les polynômes $P(x)$ à coefficients réels tels que
\[
(x + 1)P(x - 1) - (x - 1)P(x)
\]
est un polynôme constant.

OC173. On colore les entiers, chacun ayant l’une des couleurs rouge ou bleu. On sait que, pour tout ensemble fini $A$ consistant d’entiers consécutifs, la différence en valeur absolue entre les nombres d’entiers rouges et bleus dans $A$ est au plus 1000. Démontrer qu’il existe un ensemble de 2000 entiers consécutifs qui contient exactement 1000 entiers rouges et 1000 entiers bleus.

OC174. Supposons que $a$ et $b$ sont des nombres réels positifs distincts tels que $\lfloor na \rfloor$ divise $\lfloor nb \rfloor$ pour tout entier positif $n$, où $\lfloor x \rfloor$ désigne le plus grand entier inférieur ou égal à $x$. Démontrer que $a$ et $b$ sont entiers.

OC175. Supposons que $O$ est le centre du cercle circonscrit du triangle $ABC$; soit $P$ le point milieu de l’arc $BAC$ et soit $QP$ un diamètre. Soit $I$ le centre du cercle inscrit du triangle $ABC$ et soit $D$ l’intersection de $PI$ et $BC$. Le cercle circonscrit du triangle $AID$ et le prolongement de $PA$ se rencontrent au point $F$. Soit $E$ un point sur le segment $PD$ tel que $DE = DQ$. Soient $r$ et $R$ les rayons des cercles inscrit et circonscrit du triangle $ABC$ respectivement. Si $\angle AEF = \angle APE$, démontrer que
\[
\sin^2(\angle BAC) = \frac{2r}{R}.
\]
OLYMPIAD SOLUTIONS

OC111. Let \(x, y\) and \(z\) be positive real numbers. Show that
\[
x^2 + xy^2 + xyz^2 \geq 4xyz - 4.
\]

Originally question 1 from the 2012 Canadian Mathematical Olympiad.

Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; R. Hess; D. Manes; P. Perfetti; and T. Zvonaru. We give the solution used by most of the solvers.

\[
x^2 + xy^2 + xyz^2 - 4xyz + 4 = x^2 - 4x + 4 + 4x - 4xy + xy^2 + 2xy + xyz^2 - 4xy
\]
\[
= (x - 2)^2 + x(y - 2)^2 + xy(z - 2)^2 \geq 0.
\]

Editor's comment. Zvonaru remarked that we only used \(x, y \geq 0\), but \(z\) could be any real number, which is obvious from the solution.

OC112. Find all pairs of natural numbers \((a, b)\) that are not relatively prime \((\gcd(a, b) \neq 1)\) such that
\[
\gcd(a, b) + 9\text{lcm}[a, b] + 9(a + b) = 7ab.
\]

Originally question 4 from the Albanian team selection test, 2012.

Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; and K. Zelator. We give the solution similar to the solutions of Šefket Arslanagić and Konstantine Zelator (done independently).

We claim that the only solutions are \((4, 38), (38, 4)\) and \((4, 4)\).

We will use the well known equality that \(\gcd(a, b) \cdot \text{lcm}[a, b] = a \cdot b\). The equation then becomes
\[
\gcd(a, b) + 9 \frac{ab}{\gcd(a, b)} + 9(a + b) = 7ab.
\]

Let \(d := \gcd(a, b)\) and write \(a = dx, b = dy,\) with \(\gcd(x, y) = 1\). Then
\[
d + \frac{9d^2 xy}{d} + 9d(x + y) = 7d^2 xy \iff 1 + 9xy + 9(x + y) = 7dxy
\]
\[
\iff d = \frac{1 + 9xy + 9(x + y)}{7xy}.
\]

Let us now observe that if \(d \geq 5\) we have
\[
\frac{1 + 9xy + 9(x + y)}{7xy} \geq 5 \iff 26xy \leq 1 + 9x + 9y.
\]

But this is not possible, since \(9x \leq 9xy\) and \(9y \leq 9xy\) which would imply \(8xy \leq 1\).

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This shows that $d \leq 4$. Moreover, as $9xy > 7xy$ we have $d > 1$. This shows that $d \in \{2, 3, 4\}$.

If $d = 2$ we get,

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 2 \iff 5xy = 1 + 9x + 9y.$$  

Therefore,

$$y = \frac{9x + 1}{5x - 9} = 2 - \frac{x - 19}{5x - 9}.$$  

It follows that $5x - 9$ divides $x - 19$ and hence it also divides

$$(5x - 9) - 5(x - 19) = 86.$$  

This shows that $5x - 9 \in \{\pm 1, \pm 2, \pm 43, \pm 86\}$. As $x$ is an integer, the only possibilities are

$$5x - 9 = 1 \implies x = 2$$  

and

$$5x - 9 = 86 \implies x = 19.$$  

This leads to $(a, b) = (4, 38)$ and $(a, b) = (38, 4)$.

If $d = 3$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 3 \iff 12xy - 9x - 9y = 1.$$  

But this is not possible, as the left hand side is divisible by $3$, and the right hand side is not. Therefore, there is no solution in this case.

If $d = 4$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 4 \iff 19xy - 9x - 9y = 1 \iff y = \frac{9x + 1}{19x - 9}.$$  

Therefore, as $x, y \geq 1$ we have $19x - 9 \geq 0$ and

$$9x + 1 \geq 19x - 9 \implies x \leq 1.$$  

This shows that $x = 1$, and then $y = 1$. In this case we get $(a, b) = (4, 4)$.

**OC113.** Prove that among any $n$ vertices of a regular $(2n - 1)$-gon we can find 3 which form an isosceles triangle.

*Originally question 4 from Day 1 of China Western Mathematical Olympiad 2012.*

*Solved by O. Geupel; and D. Văcaru. We give the solution of Oliver Geupel.*
Our proof is by contradiction. Suppose that $V$ is a set of $n$ vertices of a regular $(2n-1)$-gon $P_1P_2\ldots P_{2n-1}$ that does not allow to form an isosceles triangle with vertices in $V$. We consider the cases $n=3$ and $n \geq 4$ in succession.

First consider the case $n=3$.

There is no loss of generality in assuming that $P_1 \in V$. Then, either $P_2$ or $P_5$ is not a member of $V$, because otherwise we would have an isosceles triangle. Also, either $P_3$ or $P_4$ is not a member of $V$. On the other hand, two points out of $P_2, P_3, P_4$, and $P_5$ must be members of $V$.

Hence, each of the pairs $\{P_2, P_3\}$ and $\{P_3, P_4\}$ contains exactly one member of $V$. Without loss of generality let $P_2 \in V$. But both $P_1P_2P_3$ and $P_1P_2P_4$ are isosceles triangles, a contradiction, which completes the case $n=3$.

It remains to consider the case $n \geq 4$.

Again suppose that $P_1 \in V$.

Then, each of the $n-1$ pairs $\{P_k, P_{2n+1-k}\}$ where $2 \leq k \leq n$, contains exactly one member of $V$. Specifically, one out of the points $P_2$ and $P_{2n-1}$ is in $V$; say $P_2 \in V$ and $P_{2n-1} \notin V$.

Then $P_3 \notin V$, because $P_1P_2P_3$ is an isosceles triangle.

Thus, $P_{2n-2} \in V$. Since $P_2 \in V$, one out of $P_4$ and $P_{2n-1}$ must be in $V$.

But we saw that $P_{2n-1} \notin V$. Therefore, $P_4$ is a member of $V$. Since $P_{2n-2} \in V$, one out of $P_{2n-3}$ and $P_{2n-1}$ must be in $V$.

But we saw that $P_{2n-1} \notin V$. Therefore, $P_{2n-3}$ is a member of $V$. Now we have obtained the isosceles triangle $P_1P_4P_{2n-3}$ in $V$, a contradiction which completes the proof.

**OC114.** Let $ABC$ be a scalene triangle. Its incircle touches $BC, AC, AB$ at $D, E, F$ respectively. Let $L, M, N$ be the symmetric points of $D, E, F$ with respect to $EF, FD$, respectively $DE$. The line $AL$ intersects $BC$ at $P$, the line $BM$ intersects $CA$ at $Q$, and the line $CN$ intersects $AB$ at $R$. Prove that $P, Q, R$ are collinear.

*Originally question 3 from China Team Selection Test 1, 2012.*

*No solutions were received for this problem.*

*Cruix Mathematicorum, Vol. 40(3), March 2014*
OC115. Find the smallest positive integer $n$ for which there exists a positive integer $k$ such that the last 2012 decimal digits of $n^k$ are all 1’s.

Originally question 4 from Brazil National Olympiad 2012. Solved by Richard Hess. We provide his solution modified by the editor.

We claim that the smallest such integer is $n = 71$.

First let us observe that for $n^k$ to end in 1, the last digit of $n$ can only be 1, 3, 7 or 9. We claim the the last digit must be 1.

Indeed, if the last digit is 3 or 7, as the order of those elements modulo 10 is four, $k$ must be a multiple of 4. Then $n^k - 1$ is divisible by $n^4 - 1$. As $n$ is odd, then $n^2 - 1$ is divisible by 4 and therefore so is $n^4 - 1$. Moreover, by Fermat Little Theorem, $n^4 - 1$ is divisible by 5.

This shows that 20 divides $n^4 - 1$, and hence it also divides $n^k - 1$. But this implies that the second last digit of $n^k$ is even, which is a contradiction.

Same way, if the last digit is 9, it follows that $k$ is even. Then $n^k - 1$ is divisible by $n^2 - 1 = (n - 1)(n + 1)$. As the last digit of $n$ is 9 we have $n + 1$ is divisible by 10 and $n - 1$ is even. This shows again that $n^2 - 1$, and therefore $n^k - 1$ is divisible by 20. Exactly as above, this is not possible.

This shows that the last digit of $n$ is one. We can then write

$$n = 10m + 1.$$ 

As $10m|n - 1|n^k - 1$, we get again $m|\frac{n^k - 1}{10} = 111.1$ and hence $m$ is odd.

As $n^k \equiv 3 \pmod 4$ it also follows that $k$ must be odd. Thus, since $n^2 \equiv 1 \pmod 8$, we have

$$7 \equiv n^k \equiv n \pmod 8.$$ 

This shows that $n \equiv 7 \pmod 8$, which shows that $n \neq 11, 51$. To complete the proof, we need to show that $n \neq 31$ and that 71 works.

Assume by contradiction that $n = 31$.

As

$$31^k \equiv 111 \equiv 31^7 \pmod{1000},$$

we get that $k - 7$ is divisible by the order of 31 modulo 1000, which is 50. Therefore $k = 50l + 7$, and then

$$31^k - 111 = 31^k - 31^7 + 31^7 = 31^7(31^k - 7 - 1) + 31^7 - 111.$$ 

Now, $31^k - 1$ is divisible by $31^{50} - 1$, and

$$31^{50} \equiv 1 \pmod{125} \quad \text{and} \quad 31^{50} \equiv 1 \pmod{16}.$$ 

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Therefore $31^{50} - 1$ is divisible by $2000$. A short computation, shows that $31^7 - 111$ is also divisible by $2000$. Hence,

$$1000 \equiv 31^k - 111 \equiv 0 \pmod{2000},$$

which is a contradiction. This proves that $n = 31$ cannot work.

We must now show that $n = 71$ works.

Let us start by observing that

$$71^{13} \equiv 31111 \pmod{10^5} \quad \text{and} \quad 71^{25} \equiv 30001 \pmod{10^5}.$$

We claim that

$$71^{10^a} \equiv 3 \cdot 10^{a+3} + 1 \pmod{10^{a+4}}.$$

We prove this by induction. For $a = 1$, it is obvious. Next we show that it being true for $a$ implies it is true for $a + 1$. We have:

$$71^{25 \times 10^a} = 3 \times 10^{a+3} + 1 + b(10^{a+4}).$$

Then

$$71^{25 \times 10^{a+1}} \equiv (3 \times 10^{a+3} + 1 + b(10^{a+4}))^{10} \pmod{10^{a+5}}$$

$$\equiv 1 + 10 \cdot (3 \times 10^{a+3} + b(10^{a+4}))$$

$$\quad + \binom{10}{2} \cdot (3 \times 10^{a+3} + b(10^{a+4}))^2 + \ldots \pmod{10^{a+5}}$$

$$\equiv 1 + 3 \times 10^{a+4} \pmod{10^{a+5}}$$

This completes the induction.

Finally, we can build recursively a sequence $r_t$ such that

$$71^{r_t} \equiv 111_\ldots 1 \pmod{10^t}.$$

Indeed, we can pick $r_5 = 13$ and then, since

$$71^{r_t} \equiv 111_\ldots 1 \pmod{10^t},$$

we get that

$$71^{r_t} - 111_\ldots 1 = 10^t \cdot x.$$ 

Since

$$71^{10^t-3} \equiv 3 \cdot 10^t + 1 \pmod{10^{t+1}},$$

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it follows from the binomial theorem that

\[ 71^{u \cdot 10^{t-3}} \equiv 3u \cdot 10^t + 1 \pmod{10^{t+1}}. \]

and hence

\[ 71^{r_t + u \cdot 10^{t-3}} \equiv (3u + x) \cdot 10^t + \underbrace{111 \ldots 1}_{t \text{ times}} \pmod{10^{t+1}}. \]

Therefore, if \( 3u + x \equiv 1 \pmod{10} \) and we define

\[ r_{t+1} = r_t + u \cdot 10^{t-3}, \]

we get

\[ 71^{r_{t+1}} \equiv \underbrace{111 \ldots 1}_{t+1 \text{ times}} \pmod{10^{t+1}}. \]