OLYMPIAD SOLUTIONS

OC111. Let $x, y$ and $z$ be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4.$$ 

*Originally question 1 from the 2012 Canadian Mathematical Olympiad.*

*Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; R. Hess; D. Manes; P. Perfetti; and T. Zvonaru. We give the solution used by most of the solvers.*

\[
x^2 + xy^2 + xyz^2 - 4xyz + 4 = x^2 - 4x + 4 + 4x - 4xy + xy^2 + 2xy + xyz^2 - 4xyz \\
= (x - 2)^2 + x(y - 2)^2 + xy(z - 2)^2 \geq 0.
\]

*Editor’s comment.* Zvonaru remarked that we only used $x, y \geq 0$, but $z$ could be any real number, which is obvious from the solution.

OC112. Find all pairs of natural numbers $(a, b)$ that are not relatively prime ($\gcd(a, b) \neq 1$) such that

$$\gcd(a, b) + 9 \text{lcm}[a, b] + 9(a + b) = 7ab.$$ 

*Originally question 4 from the Albanian team selection test, 2012.*

*Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; and K. Zelator. We give the solution similar to the solutions of Šefket Arslanagić and Konstantine Zelator (done independently).*

We claim that the only solutions are $(4, 38), (38, 4)$ and $(4, 4)$.

We will use the well known equality that $\gcd(a, b) \cdot \text{lcm}[a, b] = a \cdot b$. The equation then becomes

$$\gcd(a, b) + 9 \frac{ab}{\gcd(a, b)} + 9(a + b) = 7ab.$$ 

Let $d := \gcd(a, b)$ and write $a = dx, b = dy$, with $\gcd(x, y) = 1$. Then

\[
d + \frac{9d^2 xy}{d} + 9d(x + y) = 7d^2 xy \iff 1 + 9xy + 9(x + y) = 7dxy \\
\iff d = \frac{1 + 9xy + 9(x + y)}{7xy}.
\]

Let us now observe that if $d \geq 5$ we have

$$\frac{1 + 9xy + 9(x + y)}{7xy} \geq 5 \iff 26xy \leq 1 + 9x + 9y.$$ 

But this is not possible, since $9x \leq 9xy$ and $9y \leq 9xy$ which would imply $8xy \leq 1$. 

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This shows that $d \leq 4$. Moreover, as $9xy > 7xy$ we have $d > 1$. This shows that $d \in \{2, 3, 4\}$.

If $d = 2$ we get,

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 2 \iff 5xy = 1 + 9x + 9y.$$ 

Therefore,

$$y = \frac{9x + 1}{5x - 9} = 2 - \frac{x - 19}{5x - 9}.$$ 

It follows that $5x - 9$ divides $x - 19$ and hence it also divides

$$(5x - 9) - 5(x - 19) = 86.$$ 

This shows that $5x - 9 \in \{\pm 1, \pm 2, \pm 43, \pm 86\}$. As $x$ is an integer, the only possibilities are

$$5x - 9 = 1 \implies x = 2$$

and

$$5x - 9 = 86 \implies x = 19.$$ 

This leads to $(a, b) = (4, 38)$ and $(a, b) = (38, 4)$.

If $d = 3$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 3 \iff 12xy - 9x - 9y = 1.$$ 

But this is not possible, as the left hand side is divisible by 3, and the right hand side is not. Therefore, there is no solution in this case.

If $d = 4$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 4 \iff 19xy - 9x - 9y = 1 \iff y = \frac{9x + 1}{19x - 9}.$$ 

Therefore, as $x, y \geq 1$ we have $19x - 9 \geq 0$ and

$$9x + 1 \geq 19x - 9 \implies x \leq 1.$$ 

This shows that $x = 1$, and then $y = 1$. In this case we get $(a, b) = (4, 4)$.

**OC113.** Prove that among any $n$ vertices of a regular $(2n - 1)$-gon we can find $3$ which form an isosceles triangle.

*Originally question 4 from Day 1 of China Western Mathematical Olympiad 2012.*

*Solved by O. Geupel; and D. Văcaru. We give the solution of Oliver Geupel.*

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Our proof is by contradiction. Suppose that \( V \) is a set of \( n \) vertices of a regular \((2n−1)\)-gon \( P_1P_2\ldots P_{2n−1} \) that does not allow to form an isosceles triangle with vertices in \( V \). We consider the cases \( n = 3 \) and \( n \geq 4 \) in succession.

First consider the case \( n = 3 \).

There is no loss of generality in assuming that \( P_1 \in V \). Then, either \( P_2 \) or \( P_5 \) is not a member of \( V \), because otherwise we would have an isosceles triangle. Also, either \( P_3 \) or \( P_4 \) is not a member of \( V \). On the other hand, two points out of \( P_2, P_3, P_4, \) and \( P_5 \) must be members of \( V \).

Hence, each of the pairs \( \{P_2, P_3\} \) and \( \{P_3, P_4\} \) contains exactly one member of \( V \). Without loss of generality let \( P_2 \in V \). But both \( P_1P_2P_3 \) and \( P_1P_2P_4 \) are isosceles triangles, a contradiction, which completes the case \( n = 3 \).

It remains to consider the case \( n \geq 4 \).

Again suppose that \( P_1 \in V \).

Then, each of the \( n−1 \) pairs \( \{P_k, P_{2n+1−k}\} \) where \( 2 \leq k \leq n \), contains exactly one member of \( V \). Specifically, one out of the points \( P_2 \) and \( P_{2n−1} \) is in \( V \); say \( P_2 \in V \) and \( P_{2n−1} \notin V \).

Then \( P_3 \notin V \), because \( P_1P_2P_3 \) is an isosceles triangle.

Thus, \( P_{2n−2} \in V \). Since \( P_2 \in V \), one out of \( P_4 \) and \( P_{2n−1} \) must be in \( V \).

But we saw that \( P_{2n−1} \notin V \). Therefore, \( P_4 \) is a member of \( V \). Since \( P_{2n−2} \in V \), one out of \( P_{2n−3} \) and \( P_{2n−1} \) must be in \( V \).

But we saw that \( P_{2n−1} \notin V \). Therefore, \( P_{2n−3} \) is a member of \( V \). Now we have obtained the isosceles triangle \( P_1P_4P_{2n−3} \) in \( V \), a contradiction which completes the proof.

\textbf{OC114.} Let \( ABC \) be a scalene triangle. Its incircle touches \( BC, AC, AB \) at \( D, E, F \) respectively. Let \( L, M, N \) be the symmetric points of \( D, E, F \) with respect to \( EF, FD \), respectively \( DE \). The line \( AL \) intersects \( BC \) at \( P \), the line \( BM \) intersects \( CA \) at \( Q \), and the line \( CN \) intersects \( AB \) at \( R \). Prove that \( P, Q, R \) are collinear.

*Originally question 3 from China Team Selection Test 1, 2012.*

*No solutions were received for this problem.*

*Crux Mathematicorum, Vol. 40(3), March 2014*
Find the smallest positive integer $n$ for which there exists a positive integer $k$ such that the last 2012 decimal digits of $n^k$ are all 1’s.

*Originally question 4 from Brazil National Olympiad 2012.*

*Solved by Richard Hess. We provide his solution modified by the editor.*

We claim that the smallest such integer is $n = 71$.

First let us observe that for $n^k$ to end in 1, the last digit of $n$ can only be 1, 3, 7 or 9. We claim the the last digit must be 1.

Indeed, if the last digit is 3 or 7, as the order of those elements modulo 10 is four, $k$ must be a multiple of 4. Then $n^k - 1$ is divisible by $n^4 - 1$. As $n$ is odd, then $n^2 - 1$ is divisible by 4 and therefore so is $n^4 - 1$. Moreover, by Fermat Little Theorem, $n^4 - 1$ is divisible by 5.

This shows that 20 divides $n^4 - 1$, and hence it also divides $n^k - 1$. But this implies that the second last digit of $n^k$ is even, which is a contradiction.

Same way, if the last digit is 9, it follows that $k$ is even. Then $n^k - 1$ is divisible by $n^2 - 1 = (n - 1)(n + 1)$. As the last digit of $n$ is 9 we have $n + 1$ is divisible by 10 and $n - 1$ is even. This shows again that $n^2 - 1$, and therefore $n^k - 1$ is divisible by 20. Exactly as above, this is not possible.

This shows that the last digit of $n$ is one. We can then write

$$n = 10m + 1.$$ 

As $10m | n - 1 | n^k - 1$, we get again $m | \frac{n^k - 1}{10}$ and hence $m$ is odd.

As $n^k \equiv 3 \pmod{4}$ it also follows that $k$ must be odd. Thus, since $n^2 \equiv 1 \pmod{8}$, we have

$$7 \equiv n^k \equiv n \pmod{8}.$$ 

This shows that $n \equiv 7 \pmod{8}$, which shows that $n \neq 11, 51$. To complete the proof, we need to show that $n \neq 31$ and that 71 works.

Assume by contradiction that $n = 31$.

As

$$31^k \equiv 111 \equiv 31^7 \pmod{1000},$$

we get that $k - 7$ is divisible by the order of 31 modulo 1000, which is 50. Therefore $k = 50l + 7$, and then

$$31^k - 111 = 31^k - 31^7 + 31^7 = 31^7 (31^{k-7} - 1) + 31^7 - 111.$$ 

Now, $31^{k-7} - 1$ is divisible by $31^{50} - 1$, and

$$31^{50} \equiv 1 \pmod{125} \text{ and } 31^{50} \equiv 1 \pmod{16}.$$  

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Therefore $31^{50} - 1$ is divisible by 2000. A short computation shows that $31^7 - 111$ is also divisible by 2000. Hence,

$$1000 \equiv 31^k - 111 \equiv 0 \pmod{2000},$$

which is a contradiction. This proves that $n = 31$ cannot work.

We must now show that $n = 71$ works.

Let us start by observing that

$$71^{13} \equiv 31111 \pmod{10^5} \quad \text{and} \quad 71^{250} \equiv 30001 \pmod{10^5}.$$

We claim that

$$71^{10^a} \equiv 3 \cdot 10^{a+3} + 1 \pmod{10^{a+4}}.$$

We prove this by induction. For $a = 1$, it is obvious. Next we show that it being true for $a$ implies it is true for $a+1$. We have:

$$71^{25 \times 10^a} = 3 \times 10^{a+3} + 1 + b(10^{a+4}).$$

Then

$$71^{25 \times 10^{a+1}} \equiv (3 \times 10^{a+3} + 1 + b(10^{a+4}))^{10} \pmod{10^{a+5}}$$

$$\equiv 1 + 10 \cdot (3 \times 10^{a+3} + b(10^{a+4}))$$

$$+ \left(\frac{10}{2}\right) \cdot (3 \times 10^{a+3} + b(10^{a+4}))^2 + ... \pmod{10^{a+5}}$$

$$\equiv 1 + 3 \times 10^{a+4} \pmod{10^{a+5}}$$

This completes the induction.

Finally, we can build recursively a sequence $r_i$ such that

$$71^{r_i} \equiv \underbrace{111...1}_{t \text{ times}} \pmod{10^t}.$$

Indeed, we can pick $r_5 = 13$ and then, since

$$71^{r_i} \equiv \underbrace{111...1}_{t \text{ times}} \pmod{10^t},$$

we get that

$$71^{r_i} - \underbrace{111...1}_{t \text{ times}} \equiv 10^t \cdot x.$$

Since

$$71^{10^t-3} \equiv 3 \cdot 10^t + 1 \pmod{10^{t+1}},$$

it follows from the binomial theorem that
\[ 71^{u \cdot 10^{t-3}} \equiv 3u \cdot 10^t + 1 \pmod{10^{t+1}}. \]
and hence
\[ 71^{r^t + u \cdot 10^{t-3}} \equiv (3u + x) \cdot 10^t \underbrace{+ 111\ldots1}_{t \text{ times}} \pmod{10^{t+1}}. \]

Therefore, if \(3u + x \equiv 1 \pmod{10}\) and we define
\[ r_{t+1} = r_t + u \cdot 10^{t-3}, \]
we get
\[ 71^{r_{t+1}} \equiv \underbrace{111\ldots1}_{t+1 \text{ times}} \pmod{10^{t+1}}. \]