CONTEST CORNER

SOLUTIONS

CC61. We place 3 green, 4 yellow and 5 red balls in a bag. Two balls of different colours are selected at random, removed, and replaced with two balls of the third colour. Show that it is impossible for all of the remaining balls to be the same colour, no matter how many times this process is repeated.

*Originally problem 4c in 2004 Hypatia Contest.*

_Solved by R. Hess; I. Leifer; D. Lowry-Duda; H. Wang; and M. Wu._ We present Han Wang’s solution below.

Consider the numbers of balls modulo 3. Then the position where all remaining balls are the same colour is congruent to \((0, 0, 0)\) (mod 3), and the starting position is congruent to \((0, 1, 2)\) (mod 3). Notice that both subtracting 1 and adding 2 balls are congruent to adding 2, when considered (mod 3).

So regardless of whether we add two balls to the red, the green, or the blue coloured balls, the next position is congruent to \((2, 0, 1)\). After another move (regardless of distinguished colour), the position is congruent to \((1, 2, 0)\). After another move, the position is congruent to \((0, 1, 2)\), which is the same as the start!

So we will never reach a position congruent to \((0, 0, 0)\), and so it is impossible to get all balls to be the same colour.

CC62. For each real number \(x\), let \([x]\) be the largest integer less than or equal to \(x\). For example, \([5]\) = 5, \([7.9]\) = 7 and \([-2.4]\) = -3. An arithmetic progression of length \(k\) is a sequence \(a_1, a_2, \ldots, a_k\) with the property that there exists a real number \(b\) such that \(a_{i+1} - a_i = b\) for each \(1 \leq i \leq k - 1\). Let \(\alpha > 2\) be a given irrational number. Then \(S = \{[n\alpha] : n \in \mathbb{Z}\}\) is the set of all integers equal to \([n\alpha]\) for some integer \(n\). Prove that for any integer \(m \geq 3\), there exist \(m\) distinct numbers contained in \(S\) which form an arithmetic progression of length \(m\).

*Originally from 2013 Canadian Open Mathematics Challenge, problem C4a.*

_No solutions were received for this problem._

CC63. A quadrilateral circumscribes a circle. Prove that the perimeter of the quadrilateral bears the same ratio to the perimeter of the circle as the area of the quadrilateral bears to the area of the circle.

*Originally 1976 Descartes Contest, problem 4.*

_Solved by S. Arslanagić; N. Evgenidis; R. Hess; E. H. Pilehrood; N. Stanciu and T. Zvonaru._ We give a solution based on all submitted solutions.

_Crux Mathematicorum, Vol. 40(3), March 2014_
Let $ABCD$ be the quadrilateral circumscribing the circle centred at $O$; let $AB = a, BC = b, CD = c$ and $AD = d$. Let $P_c$ and $P_q$ be the respective perimeters of the circle and quadrilateral, and let $A_c$ and $A_q$ be the respective areas of the circle and quadrilateral.

We have $P_q = a + b + c + d$, where $a, b, c,$ and $d$ are the side-lengths of the quadrilateral, and $P_c = 2\pi r$, where $r$ is the radius of the circle.

The area of the quadrilateral is the sum of the areas of the four triangles $ABO$, $BCO$, $CDO$, $ADO$, all of which have height $r$, since the radii are at right angles to the sides of the quadrilateral. Thus

$$A_q = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} + \frac{dr}{2}$$

$$= \frac{1}{2}(a + b + c + d)r.$$

Since the area of the circle is $A_c = \pi r^2$, it follows that

$$\frac{P_q}{P_c} = \frac{a + b + c + d}{2\pi r} = \frac{(a + b + c + d)r}{2\pi r^2} = \frac{A_q}{A_c}.$$  

**CC64.** Show that a power of 2 can never be the sum of $k$ consecutive positive integers, $k > 1$.

*Originally problem 9 from 1978 Descartes Contest.*

_Solved by M. Bataille; M. Coiculescu; I. D. Gerganov; R. Hess; D. Manes; E. H. Pilehrood; V. Văcaru; E. Wang; T. Zvonaru and N. Stanciu. We present the solution by Matei Coiculescu._

Suppose there exist positive integers $k > 1, n$, and $m$ such that

$$2^n = m + (m + 1) + \cdots + (m + k - 1) = mk + \frac{k(k - 1)}{2}$$

This gives

$$2^{n+1} = k(2m + k - 1)$$

If $k$ is even, then $(2m + k - 1)$ is odd and greater than 1, which is not possible as $2^{n+1}$ has only even factors. It is also not possible for $k$ to be odd for the same reason. This is a contradiction, hence a power of 2 can never be the sum of $k > 1$ positive integers.

**CC65.** Suppose that three circles in the plane are located so that each pair of circles intersect in two points, thereby giving a common chord to those two circles. Prove that these three chords pass through one point.

*Originally question 8 from 2004 APICS Math Competition.*

Copyright © Canadian Mathematical Society, 2015
Solved by G. Tsapakidis; and N. Stanciu and T. Zvonaru. We present a solution based on the one by George Tsapakidis.

Let $C_1, C_2, C_3$ be three circles such that $C_1$ intersects $C_3$ at points $A$ and $B$; $C_1$ intersects $C_2$ at points $C$ and $D$; and, $C_2$ intersects $C_3$ at points $E$ and $F$. Let $P$ be the intersection point of the chords $AB$ and $CD$. If $E = P$, then we are done, so suppose $E \neq P$. By the Theorem of Intersecting Chords, we have

$$|AP| \cdot |PB| = |CP| \cdot |PD|,$$

(1)

Suppose that a line containing the segment $EP$ does not contain the chord $EF$, so that it intersects circles $C_2$ and $C_3$ at the points $L$ and $K$, respectively, where $F$, $K$ and $L$ are all distinct. Now $EK$ and $AB$ are intersecting chords of $C_3$, while $EL$ and $CD$ are intersecting chords of $C_2$. By the Theorem of Intersecting Chords, we now have

$$|EP| \cdot |PK| = |AP| \cdot |PB|,$$

(2)

and

$$|EP| \cdot |PL| = |CP| \cdot |PD|.$$  

(3)

Combining Equations (1–3), we obtain $|EP| \cdot |PL| = |EP| \cdot |PK|$. We have $|EP| > 0$, since $E \neq P$, therefore $|PK| = |PL|$, and since the segments $PK$ and $PL$ lie on the same line, we must have $K = L$. This is a contradiction, so it must be the case that the chord $EF$ also passes through $P$.

Editor’s comments. Victor Pambuccian has noted that this problem appears in Hilbert’s lectures on the foundations of geometry, going back to 1896. It is true under significantly weaker assumptions than the axiom system for the standard Euclidean plane. For specifics, see V. Pambuccian’s review of David Hilbert’s lectures on the foundations of geometry, 1891–1902, in Philos. Math. (3) 21 (2013), no. 2, 255–277.

There is a subtle oversight in the wording of the original problem; the chords need
not intersect at all! No counterexample was submitted, perhaps due to the power of suggestion, so we state a very natural modification of the problem:

Suppose that three circles in the plane are located so that each pair of circles intersect in two points, thereby giving a common chord to those two circles. Prove that if two of these chords intersect, then the third chord also intersects them at the same point.

Thus, we avoid the cases where all three chords are contained in mutually parallel secant lines, or where the secants intersect outside the three circles.

---

Math Quotes

The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up.

The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a recipriversexclusion, a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipverseclusions now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else’s Problem field.

The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.)