Chebyshev polynomials and recursive relations (II)

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Continued from ***Crux***, Volume 40 (8). In the first part of this article, we defined Chebyshev polynomials of the first and second type and exhibited some of their properties. Here we continue this discussion. Recall the following numbered equations from part I of this article:

$$P_{n+1}(x) = x \cdot P_n(x) - P_{n-1}(x), \quad \text{where} \quad P_0(x) = 1, \ P_1(x) = x; \tag{1}$$

$$P_n(2\cos\phi) = \frac{\sin((n+1)\phi)}{\sin\phi}; \tag{2}$$

for polynomials $Q_i(x)$ with $Q_0(x) = 2, Q_1(x) = x$ that satisfy (1) we have:

$$2\cos n\phi = Q_n(2\cos\phi); \tag{2$'$}$$

$$P_n(x) = \frac{(x+\sqrt{x^2-4})^{n+1} - (x-\sqrt{x^2-4})^{n+1}}{2^{n+1}\sqrt{x^2-4}}. \tag{3}$$

3 Polynomial roots and various products

Many interesting problems that involve symmetrical combinations of n numbers (or letters) can be easily solved if these numbers are viewed as roots of some polynomials of degree n. Given n numbers $\gamma_k = 2\cos\frac{k\pi}{n+1}$, for $k = 1, 2, \ldots, n$, the corresponding polynomial is the aforementioned $P_n(x)$: indeed, if you substitute $\phi = \frac{\pi}{n+1}, \frac{2\pi}{n+1}, \ldots, \frac{n\pi}{n+1}$ into (2), we see that $\gamma_n = 2\cos\frac{k\pi}{n+1}$ are the roots of $P_n(x)$. Here we need the following well-known result: if γ is a root of a polynomial $F(x)$, then $F(x)$ is divisible by $x - \gamma$. Therefore, the polynomial $P(x)$ is divisible by all the polynomials $x - \gamma_k$ and hence is divisible by their product. Since $P_n(x)$ is of degree n and has leading coefficient equal to 1, it is equal to the product $\prod_{1\le k\le n}(x - \gamma_k)$. So we have:

$$P_n(x) = \prod_{1\le k\le n}\left(x - 2\cos\frac{k\pi}{n+1}\right). \tag{4}$$

Exercise 5.

a) Prove the identity $Q_n(x) = \prod_{1\le k\le n}\left(x - 2\cos\frac{(2k-1)\pi}{2n}\right)$ (4$'$)

b) Check (4) and (4$'$) for $n = 2, 3, 4, 5$.

Next, we will derive one curious identity, which follows from combining (2) and (4). Let us calculate $P_{2m}(0)$ for $m > 0, m \in \mathbb{Z}$ in two different ways and equate the results. On the one hand, from (2) we have:

$$P_{2m}(0) = P_{2m}\left(2\cos\frac{\pi}{2}\right) = \left(\sin\frac{(2m+1)\pi}{2}\right) / \sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + m\pi\right) = (-1)^m.$$

On the other hand, from (4) we get:

$$P_{2m}(0) = \prod_{1\le k\le 2m}\left(-2\cos\frac{k\pi}{2m+1}\right).$$

For $m+1 \le k \le 2m$, replace each $\cos\frac{k\pi}{2m+1}$ by $-\cos\left(\pi - \frac{k\pi}{2m+1}\right)$ to get

$$P_{2m}(0) = (-1)^m\left[2^m \prod_{1\le k\le m}\cos\frac{k\pi}{2m+1}\right]^2.$$

The expression in square brackets is positive because it involves only cosines of acute angles; it is therefore equal to 1 (which is easily seen after equating the two expressions for $P_{2m}(0)$), that is

$$\prod_{1\le k\le m}\cos\frac{k\pi}{2m+1} = \frac{1}{2^m}. \tag{5}$$

In words, (5) can be described as follows: for $m > 0$, the geometric mean of cosines of acute angles that are multiples of $\frac{\pi}{2m+1}$ is equal to $\frac{1}{2}$.

Exercise 6.

a) Find $P_n(1), P_n(-1), Q_n(1), Q_n(-1)$.

Prove the following identities similar to (5):

b) $\prod_{1\le k\le m}\sin\dfrac{k\pi}{2m} = \dfrac{\sqrt{m}}{2^{m-1}}, \ \ m \ge 1;$

c) $\prod_{1\le k\le m}\tan\dfrac{k\pi}{2m+1} = \sqrt{2m+1}, \ \ m \ge 1;$

d) $\prod_{1\le k\le m}\cos\dfrac{(2k-1)\pi}{4m} = \dfrac{\sqrt{2}}{2^m}, \ \ m \ge 1.$

Exercise 7. Determine for which values of m and n

a) polynomial P_n is divisible by polynomial P_m;

b) polynomial Q_n is divisible by polynomial Q_m.

4 Generating functions, series and coefficients

In this section, we will explore a method that is useful in many areas of mathematics — analysis, combinatorics, probability theory — the method of *generating functions*. This method allows us to use separate elements of a sequence to receive information about the whole sequence.

For a sequence $a_0, a_1, a_2, \ldots$, its generating function is given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots = \sum_{n \geq 0} a_n z^n.$$

These expressions are also known as *power series*. You can add, subtract and multiply power series like any other polynomials, you can divide one power series by another (if the divisor's constant term is not equal to 0), you can differentiate and integrate the whole series term by term; in fact, you can use all of these operations to get new series. It is often possible to use a recurrence relation definition of the function to find a simple formula for its generating function and vice versa — use the generating function to find a formula or a relation for elements of the sequence.

For a finite sequence $a_0, a_1, \ldots a_n$, its generating function is polynomial $f(z) = a_0 + a_1 z + \ldots + a^n z^n$. For example, the polynomial $f_n(z) = (1+z)^n$ is a generating function for the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, which appear in the nth row of Pascal's triangle (Table 2 in Part I of this article):

$$\sum_{0 \leq k \leq n} \binom{n}{k} z^k = (1+z)^n. \tag{6}$$

Differentiating (6) enough times and then setting $z = 0$, we find that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot \cdots \cdot k}.$$

Consider the obvious identity $(1+z)(1+z)^n = (1+z)^{n+1}$ written as

$$(1+z)\left(\sum_{0 \leq k \leq n} \binom{n}{k} z^k\right) = \sum_{0 \leq k \leq n+1} \binom{n+1}{k} z^k.$$

Opening the brackets and considering the coefficients of z^m on both sides of the above expression, we get an important identity

$$\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}.$$

Amongst all infinite series, geometric series have a particularly nice closed form of the power series. Let $b_0 = b$ and $b_n = q b_{n-1}$. Then replace each b_n by $q b_{n-1}$ in the sum

$$f(z) = \sum_{n \geq 0} b_n z^n = b + \sum_{n \geq 1} b_n z^n$$

to get

$$f(z) = b + qz \sum_{n \geq 1} b_{n-1} z^{n-1} = b + qzf(z),$$

from which we have $f(z)(1 - qz) = b$, and hence

$$f(z) = \sum_{n \geq 0} b_n z^n = \frac{b}{1 - qz}. \tag{7}$$

This is, of course, the well-known formula for the sum of a geometric series for $|qz| < 1$. But using this method, we can also get the generating function for our sequence of polynomials $P_n(x)$.

Let $\Phi(z) = \sum_{n \geq 0} P_n(x) z^n = 1 + xz + \sum_{n \geq 2} P_n(x) z^n$. Here x is a parameter and below, for ease of notation, we will use P_i instead of $P_i(x)$. By (1), replace each P_n (for $n \geq 2$) by $xP_{n-1} - P_{n-2}$. Then

$$\begin{aligned}
\Phi(z) &= 1 + xz + \sum_{n \geq 2} xP_{n-1} z^n - \sum_{n \geq 2} P_{n-2} z^n \\
&= 1 + xz + xz \sum_{n \geq 2} P_{n-1} z^{n-1} - z^2 \sum_{n \geq 2} P_{n-2} z^{n-2} \\
&= 1 + xz + xz(\Phi(z) - 1) - z^2 \Phi(z).
\end{aligned}$$

Therefore, $\Phi(z) \cdot (z^2 - xz + 1) = 1$ so

$$\Phi(z) = \frac{1}{z^2 - xz + 1}. \tag{8}$$

This simple formula contains the entire sequence of polynomials P_n that we have been studying! We can obtain separate P_n, that hide within it, in two different ways.

Method 1. For $|x| > 2$, the quadratic equation $z^2 - xz + 1 = 0$ has two roots:

$$u = \frac{x + \sqrt{x^2 - 4}}{2}, \quad v = \frac{x - \sqrt{x^2 - 4}}{2}. \tag{9}$$

Since $z^2 - xz + 1 = (z - u)(z - v)$ and $uv = 1$, we get:

$$\Phi(z) = \frac{1}{(z - u)(z - v)} = \left(\frac{u}{1 - zu} - \frac{v}{1 - zv} \right) \frac{1}{u - v} = \sum_{n \geq 0} \frac{u^{n+1} - v^{n+1}}{u - v} z^n,$$

that is $P_n(x) = \dfrac{u^{n+1} - v^{n+1}}{u - v}$, which is formula (3).

Method 2. Let us use (8) to find separate coefficients for each polynomial $P_n(x)$. Here is how:

$$\begin{aligned}
\Phi(z) &= \frac{1}{1 - (xz - z^2)} = \sum_{k \geq 0} (xz - z^2)^k \\
&= \sum_{k \geq 0} \left(\sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} x^{k-i} z^{k+i} \right) = \sum_{n \geq 0} z^n \left(\sum_i (-1)^i \binom{n-i}{i} x^{n-2i} \right),
\end{aligned}$$

where we used (7), (6) and then separated the coefficient of z^n, which is the needed $P_n(x)$. So we have:

$$P_n(x) = \sum_i (-1)^i \binom{n-i}{i} x^{n-2i}. \tag{10}$$

For example, $P_6(x) = \binom{6}{0}x^6 - \binom{5}{1}x^4 + \binom{4}{2}x^2 - \binom{3}{3} = x^6 - 5x^4 + 6x^2 - 1.$

Of course, one can prove the existing formulas (3) and (10) without generating functions; but the most exciting thing is how they appear, very easily, from the simple formula (8).

Note: we could perform all the operations with the power series above because for small values of z all the above series converge (for instance, (7) converges for $|z| < 1/|q|$).

Exercise 8. Consider the Fibonacci sequence given by

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = u_n + u_{n-1}.$$

a) Prove that its generating function is equal to $\dfrac{z}{1-z-z^2}$.

b) Deduce that $u_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$.

c) Prove the identity $u_n = \sum_i \binom{n-i-1}{i}$.

Exercise 9.

a) Find the generating function for the sequence of polynomials $Q_n(x)$ define at the beginning of this article. Use it to show that for $|x| > 2$ we have

$$Q_n(x) = u^n - v^n,$$

where u and v are given by (9). Note: this is exercise 3c.

b) Show that formulas (3) and (3′) for $|x| < 2$ turn into formulas (2) and (2′). (Hint: $u = \cos\phi + i\sin\phi, v = \cos\phi - i\sin\phi$ if $x = 2\cos\phi$.)

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