SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


3891. Proposed by Michel Bataille.

Let \( ABC \) be a triangle that is neither equilateral nor right-angled and let \( O \) be its circumcentre. Let the altitudes from \( A, B \) and \( C \) meet the circumcircle again at \( A', B' \) and \( C' \), respectively. If \( U, V \) and \( W \) denote the circumcentres of triangles \( OBC, OCA \) and \( OAB \), respectively, prove that the lines \( UA', VB' \) and \( WC' \) are concurrent and identify their common point.

We received seven solutions, all correct. We present four of these solutions.

Solution 1, by the proposer.

Let \( I \) be the inversion in the circumcircle \( \Gamma \) of triangle \( ABC \) and \( J \) be the inversion in the circumcircle \( \Delta \) of triangle \( OBC \). Let \( H \) be the orthocentre of triangle \( ABC \). The points \( H \) and \( A' \) are reflections of each other in the axis \( BC \); any circle through these two points is orthogonal to \( BC \). The inversion \( I \) fixes \( A' \) and carries the line \( BC \) to \( \Delta \). Let \( P = I(H) \). Then, because \( I \) preserves orthogonality, any circle through \( A' \) and \( P \) is orthogonal to \( \Delta \), so that \( J(A') = P \). Therefore \( A'P \) passes through the centre \( U \) of \( \Delta \). [See R.A. Johnson, Advanced Euclidean Geometry, Houghton-Mifflin, 1929, reprinted by Dover, 1960, paragraph 80, p. 55.] Thus \( UA' \), and similarly \( VB' \) and \( WC' \), all pass through the point \( P = I(H) \).

Solution 2, by Oliver Geupel (slightly modified by the editor).

We prove that the three lines are concurrent at the inverse point \( P \) of the orthocentre \( H \) with respect to inversion in the circumcircle of the triangle \( ABC \).

Consider the problem in the complex plane where the circumcircle is the unit circle and the geometric inverse of \( z \) is \( 1/\bar{z} \). Let \( a, a', b, c, h, p, u \) denote the affixes of the respective points \( A, A', B, C, H, P, U \). Noting that two vectors \( z \) and \( w \) are perpendicular if and only if \( z/\bar{z} = -w/\bar{w} \), we deduce from \( AA' \perp BC \) that

\[
-a' = \frac{a - a'}{(1/a) - (1/a')} = \frac{a - a'}{\bar{a} - a'} \quad \text{and} \quad -\frac{b - c}{b - c} = \frac{-\frac{b - c}{(1/b) - (1/c)}}{bc} = bc,
\]

whence

\[
a' = -\frac{bc}{a} \quad \text{and} \quad \bar{a}' = -\frac{a}{bc}.
\]

The condition \( uu = (u - b)(\bar{u} - b) \) leads to \( \bar{b}u + b\bar{u} = 1 \). Similarly \( \bar{c}u + c\bar{u} = 1 \). From this, we find that

\[
u = \frac{bc}{b + c} \quad \text{and} \quad \bar{u} = \frac{1}{b + c}.
\]

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Since \( h = a + b + c \) and \( h \bar{p} = 1 \), 
\[
\bar{p} = \frac{1}{a + b + c} \quad \text{and} \quad p = \frac{abc}{ab + bc + ca}.
\]

A straightforward computation yields 
\[
(a' - u)(\bar{a'} - \bar{p}) = \left( \frac{bc}{a} + \frac{bc}{b + c} \right) \left( \frac{a}{bc} + \frac{1}{a + b + c} \right) = \frac{a^2 + ab + bc + ca}{a(b + c)}
\]
\[
= \left( \frac{a}{bc} + \frac{1}{b + c} \right) \left( \frac{bc}{a} + \frac{abc}{ab + bc + ca} \right) = (\bar{a} - \bar{u})(\bar{a} - p).
\]

Hence the complex number \((a' - u)/(a' - p)\), being equal to its conjugate, is real. Thus the line \(A'U\) passes through the point \(P\). Similarly, the lines \(B'V\) and \(C'W\) pass through \(P\). This completes the proof.

Solution 3, by Titu Zvonaru.

Conventionally, let \(a, b, c, R\) and \(H\) be the sides, circumradius and orthocentre of triangle \(ABC\). Suppose \(D\) on \(BC\) is the foot of the altitude from \(A\). It is known that \(HD = DA' = 2R\cos B\cos C\). Since \(BC\) subtends the angle \(2A\) at \(O\) and the angle \(180^\circ - 2A\) at any point on the arc of the circumcircle of \(OBC\) opposite \(O\), it follows that 
\[
OU = BU = \frac{a}{2\sin 2A} = \frac{2R\sin A}{4\sin A\cos A} = \frac{R}{2\cos A}.
\]

Let \(UA'\) and \(OH\) intersect at \(P\). Because \(OU \parallel HA'\), 
\[
\frac{PO}{PH} = \frac{OU}{HA'} = \frac{R}{2\cos A} \cdot \frac{1}{4R\cos B\cos C} = \frac{1}{8\cos A\cos B\cos C}.
\]

Therefore, \(UA', VB'\) and \(WC'\) are concurrent at \(P\), which lies on the Euler line \(OH\) and satisfies 
\[
\frac{PO}{PH} = \frac{1}{8\cos A\cos B\cos C}.
\]

(This argument can be adapted if one angle is obtuse.)

Solution 4, by Prithwijit De.

We assume that the triangle is acute and not equilateral. Let \(X = UW \cap OB\) and \(Y = UV \cap OC\). Since \(UW\) right bisects \(OB\), \(\angle UXO = 90^\circ\). Similarly, \(\angle UYO = 90^\circ\), so that the quadrilateral \(UXOY\) is concyclic. Therefore, 
\[
\angle WUV = \angle XUY = 180^\circ - \angle XOY = 180^\circ - \angle BOC = 180^\circ - 2A = \angle C'A'B' .
\]

To see the last equality, note that 
\[
\angle AA'C' = \angle ACC' = 90^\circ - \angle BAC = \angle ABB' = \angle AA'B.
\]
(so that the line \(AA'\) bisects \(\angle B'A'C'\)). Similarly, \(\angle VWU = \angle B'C'A'\) and \(\angle UVW = \angle A'B'C'\). Thus triangles \(A'B'C'\) and \(UVW\) are directly similar, and the lines \(UA', VB'\) and \(WC'\) concur at the centre \(P\) of homothety.

Since the line \(AA'\) bisects \(\angle B'A'C'\), and, similarly, \(BB'\) and \(CC'\) bisect respectively \(\angle A'B'C'\) and \(\angle A'C'B'\), the incentre of triangle \(A'B'C'\) is the orthocentre \(H\) of triangle \(ABC\).

Since \(O\) is the midpoint of the arc \(BC\) of the circumcircle with centre \(U\) of triangle \(OBC\), \(\angle OUB = \angle OUC\), and \(OU\) bisects \(\angle WUV\). Similarly \(OV\) bisects \(\angle UVW\) and \(OW\) bisects \(\angle UWV\). Therefore the incentre of triangle \(UVW\) is \(O\). The line joining the incentres of triangles \(A'B'C'\) and \(UVW\) passes through the centre of homothety, so that \(O, H\) and \(P\) are collinear.

From solution 2, note that \(A'B'/UV = A'H/UO = 8 \cos A \cos B \cos C\). Also

\[
OH^2 = \left(\frac{a}{2} - c \cos B\right)^2 + (2R \cos B \cos C - R \cos A)^2
\]

\[
= R^2[(\sin A - 2 \sin C \cos B)^2 + (2 \cos B \cos C - \cos A)^2]
\]

\[
= R^2[1 - 4 \sin A \sin C \cos B + 4 \cos^2 B - 4 \cos A \cos B \cos C]
\]

\[
= R^2[1 - 4 \cos B(\cos A \cos C + \sin A \sin C + \cos(A + C))]
\]

\[
= R^2[1 - 8 \cos A \cos B \cos C].
\]

Therefore,

\[
1 - \left(\frac{OH}{R}\right)^2 = 8 \cos A \cos B \cos C = \frac{A'B'}{UV} = \frac{PH}{PO}.
\]

This simplifies to \(OH \cdot OP = R^2\), from which we see that \(P\) is the inversion of \(H\) in the circumcircle of triangle \(ABC\).

**3892. Proposed by George Apostolopoulos.**

Let \(a, b\) and \(c\) be the lengths of the sides of a triangle \(ABC\) with circumradius \(R\) and inradius \(r\). Prove that

\[
\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3 + b^3 + c^3}{3abc}.
\]

We received 11 correct solutions and one incorrect solution. We present a composite of similar solutions by Šefket Arslanagić, Michel Bataille, Scott Brown, John Huer, Kee-Wai Lau, Dragoljub Milošević, and Titu Zvonaru.

Using the identities

\[
\cos A + \cos B + \cos C = \frac{R + r}{R},
\]

\[
a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)
\]

and

\[abc = 4Rrs,\]

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[for example in D. S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989], the claimed inequality is equivalent to

\[
\frac{R}{r} \geq \frac{2}{3} \left( \frac{R + r}{R} \right) + \frac{2s(s^2 - 3r^2 - 6Rr)}{12Rrs}.
\]

This can be rewritten as \(6R^2 + 2Rr - r^2 \geq s^2\). Gerretsen’s Inequality asserts that

\[
4R^2 + 4Rr + 3r^2 \geq s^2,
\]

so it suffices to show that

\[
6R^2 + 2Rr - r^2 \geq 4R^2 + 4Rr + 3r^2.
\]

But this is equivalent to

\[
0 \leq 2R^2 - 2Rr - 4r^2 = 2(R - 2r)(R + r),
\]

which holds by Euler’s inequality \(R \geq 2r\).

3893. *Proposed by Ovidiu Furdui.*

Let \(n \geq 1\) be an integer and let the decimal part of a real number \(a\) be defined by \(\{a\} = a - \lfloor a \rfloor\). Evaluate

\[
\int_0^{\pi/2} \sin 2x \{\ln^{n-1} \tan x\} \, dx.
\]

We received six correct solutions. We present the solution of the AN-anduud Problem Solving Group.

With the substitution \(u = \tan x\), the integral becomes

\[
I = \int_0^1 \frac{2u}{(1 + u^2)^2} \{\log^{n-1} u\} \, du + \int_1^\infty \frac{2u}{(1 + u^2)^2} \{\log^{n-1} u\} \, du.
\]

If in this first integral we substitute \(v = \frac{1}{u}\), we obtain

\[
I = \int_1^\infty \frac{2v}{(1 + v^2)^2} \{-\log^{n-1} v\} \, dv + \int_1^\infty \frac{2u}{(1 + u^2)^2} \{\log^{n-1} u\} \, du
\]

\[
= \int_1^\infty \frac{2x}{(1 + x^2)^2} \{(-\log^{n-1} x) + \{\log^{n-1} x\}\} \, dx
\]

\[
= \int_1^\infty \frac{2x}{(1 + x^2)^2} \, dx = \frac{1}{2},
\]

using the equation \(\{y\} + \{-y\} = 1\), which holds for all \(y \in \mathbb{R} \setminus \mathbb{Z}\).

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3894. Proposed by Paul Bracken.
a) Prove that for \( n \in \mathbb{N} \),

\[
1 + 2 \sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{n+1} \frac{1}{k+n}.
\]

b) For the equation in part a), evaluate the limit of the right-hand side as \( n \to \infty \) and compute the sum in closed form.

We received 12 correct submissions. We present two solutions for part a, and three solutions for part b, each utilized by multiple solvers.

For ease of solution-wide notation, let \( S_n \) be the left-hand-side of the identity, and let \( T_n \) be the right-hand-side.

Part a, Solution 1. It is easily checked that

\[
1 + 2 \sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{n} \frac{1}{3k+1} + \sum_{k=1}^{n} \frac{1}{3k-1} - 2 \sum_{k=1}^{n} \frac{1}{3k}
\]

\[
= \left( \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{3k} \right) - 2 \sum_{k=1}^{n} \frac{1}{3k}
\]

\[
= \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{n+1} \frac{1}{k}
\]

that is,

\[
1 + 2 \sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{n+1} \frac{1}{k+n}.
\]

Part a, Solution 2. It is simple to check that for \( n = 1 \), both sides of the proposed equality evaluate to \( \frac{13}{12} \). After assuming that the identity is true for \( n \), in order to prove that it is also true for \( n+1 \) it is enough to prove that \( S_{n+1} - S_n = T_{n+1} - T_n \), for then we may add \( S_n = T_n \) to both sides to obtain the result. So we must prove

\[
\frac{2}{(3(n+1))^3 - 3(n+1)} = \sum_{k=1}^{n+3} \frac{1}{k+n+1} - 2 \sum_{k=1}^{n+1} \frac{1}{k+n}.
\]

But the right-hand side of the last equation is

\[
\frac{1}{3n+4} + \frac{1}{3n+3} + \frac{1}{3n+2} - \frac{1}{n+1} = \frac{2}{(3(n+1))^3 - 3(n+1)}.
\]

Part b, Solution 1. From the above calculation in part a, the sum \( T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n} \) satisfies

\[
T_n = \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = H_{3n+1} - H_n
\]
where as usual $H_m$ denotes $\sum_{k=1}^{m} \frac{1}{k}$. It is well-known that for all positive integer $m$, we have $H_m = \ln(m) + \gamma + \varepsilon(m)$ where $\gamma$ is the Euler-Mascheroni constant and $\lim_{m \to \infty} \varepsilon(m) = 0$. As a result, we may write

$$T_n = \ln(3n+1) + \gamma + \varepsilon(3n+1) - \ln(n) - \gamma - \varepsilon(n) = \ln \left( \frac{3n+1}{n} \right) + \varepsilon(3n+1) - \varepsilon(n)$$

and so

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \ln \left( 3 + \frac{1}{n} \right) = \ln(3).$$

**Part b, Solution 2.** The right-hand side of the identity is bounded:

$$\int_{n+1}^{3n+2} \frac{1}{x} \, dx \leq \sum_{k=n+1}^{3n+1} \frac{1}{k} \leq \int_{n}^{3n+1} \frac{1}{x} \, dx.$$

Evaluating both sides of the inequality gives

$$\ln \left( \frac{3n+2}{n+1} \right) \leq \sum_{k=n+1}^{3n+1} \frac{1}{k} \leq \ln \left( \frac{3n+1}{n} \right),$$

and taking the limit as $n$ tends to infinity yields

$$1 + 2 \sum_{k=1}^{\infty} \frac{1}{(3k)^3 - 3k} = \lim_{n \to \infty} T_n = \ln(3).$$

**Part b, Solution 3.** Observe that we have:

$$T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{k+n} + \sum_{k=n+1}^{2n} \frac{1}{k+n} + \frac{1}{3n+1}$$

$$= \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} + \sum_{k=1}^{n} \frac{1}{2 + \frac{k}{n}} + \frac{1}{3n+1}.$$

The two sums in the last line are Riemann sums for integrals of $\frac{1}{1+x}$ and $\frac{1}{2+x}$ over $[0,1]$, respectively, with constant step size equal to $\frac{1}{n}$, and right-hand endpoints. Taking $n$ to infinity gives:

$$\lim_{n \to \infty} T_n = \int_{0}^{1} \frac{1}{1+x} \, dx + \int_{0}^{1} \frac{1}{2+x} \, dx + \lim_{n \to \infty} \frac{1}{3n+1}$$

$$= \ln(2) - \ln(1) + \ln(3) - \ln(2) + 0 = \ln(3).$$

**Editor’s Comments.** All solvers basically used some combination of these solutions. The induction proof in Part 1, Solution 2 is a difference equation technique. The Riemann sum argument is clever, but needs careful treating; that is, $T_n$ itself is not a Riemann sum, but it is a Riemann sum plus a term that goes to 0. Finally,
the comment about “a closed form” was, I believe, intended to mean “compute the value of the limit of this expression”, not to find a closed form expression for the finite sum $T_n$.

3895. Proposed by Neculai Stanciu and Titu Zvonaru.

In the acute triangle $ABC$ with $AB \neq AC$, let $A'$ be the foot of the altitude from $A$, and let the bisector of the angle at $A$ meet $BC$ at $D$ and the circumcircle at $M$. Finally, for a point $T$ on the segment $AD$, let $P$ and $N$ be its projections on $AA'$ and $BC$, respectively. Prove that if $M, N$, and $P$ are collinear, then $T$ is the incentre of the triangle.

Seven solutions were submitted, all correct. We give two solutions.

Solution 1, by Michel Bataille.

We first establish that $T$ is the incentre of triangle $ABC$ if and only if $MB = MT$.

For, $T$ is the incentre $\iff$ it lies on the bisector of angle $B \iff \angle CB T = \angle AB T \iff \angle T B M - \angle C B M = \angle B T M - \angle C B M \iff \angle T B M = \angle B T M \iff MB = MT$.

Suppose that $M, N, P$ are collinear and that $U$ is the midpoint of $BC$. Wolog, let $b = AC > AB = c$ so that $DC > DB$ (since $DC : DB = b : c$). Since $BU/BM = \cos(\angle MBC) = \cos(\angle MAC) = \cos(A/2)$, it follows that $MB = a/(2 \cos A/2)$, where $a = BC$. Let $M'$ be the point diametrically opposite to $M$ on the circumcircle of triangle $ABC$. Since $\angle MAM' = 90^\circ$ and

$$\angle AM M' = 90^\circ - \angle AM' M = 90^\circ - \angle ACM = 90^\circ - \left(\frac{A}{2}\right) = \frac{B - C}{2},$$

we have that

$$MA = MM' \cos(\angle AM M') = 2R \cos \frac{B - C}{2}.$$

Also

$$\frac{a}{2} \tan \frac{A}{2} = BU \tan \frac{A}{2} = MU = MD \cos \frac{B - C}{2},$$

whereupon

$$MD = \frac{a \tan A/2}{2 \cos((B - C)/2)}.$$

Since $PT \parallel ND$ and $AP \parallel NT$, the homothety with centre $M$ that takes $N \rightarrow P$ also takes $D \rightarrow T$ and $T \rightarrow A$. Therefore $MA : MT = MT : MD$, and so

$$MT^2 = MA \cdot MD = 2R \cos \frac{B - C}{2} \cdot \frac{a \tan A/2}{2 \cos((B - C)/2)}$$

$$= aR \tan \frac{A}{2} = \frac{a^2}{2 \sin A} \tan \frac{A}{2} = \frac{a^2}{4 \cos^2(A/2)} = MB^2.$$

Thus $MT = MB$ and the desired result follows.
Solution 2, by Madhav R. Modak.

Let $AB < AC$, so that the points on BC are in the order $BA'NDC$. Let $AT = x$ and $TD = y$. Since $M$, $N$ and $P$ are collinear, Menelaus’ theorem for triangle $AA'D$ and transversal $PNM$ yields

\[ \frac{A'N}{ND} \cdot \frac{DM}{MA} \cdot \frac{AP}{PA'} = -1. \] (1)

Since $AA' \parallel TN$ and $PT \parallel A'D$, we have that $AP/PA' = A'N/ND = x/y$, so (1) gives numerically,

\[ \frac{DM}{MA} = \frac{y^2}{x^2}. \] (2)

Since $\angle DBM = \frac{1}{2} \angle CAM = \frac{1}{2}A$ and $\angle BMD = \angle BCA = C$, the Sine Law applied to triangle $BDM$ give $BD/\sin C = DM/\sin \frac{1}{2}A$. Since $\angle ABM = B + \frac{1}{2}A$ and $\angle BMA = C$, the Sine Law applied to triangle $ABM$ gives $MA/\sin(B + \frac{1}{2}A) = AB/\sin C$. These give

\[ \frac{DM}{MA} = \frac{BD}{AB} \cdot \frac{\sin \frac{1}{2}A}{\sin(B + \frac{1}{2}A)}. \] (3)

In triangle $ABD$, $\angle ABD = 180^\circ - (B + \frac{1}{2}A)$ and $\angle BAD = \frac{1}{2}A$, so that, by the Sine Law, $BD/\sin \frac{1}{2}A = AB/\sin(B + \frac{1}{2}A)$. With (2) and (3), this yields that

\[ \frac{y^2}{x^2} = \frac{DM}{MA} = \left( \frac{BD}{AB} \right)^2 \quad \text{or} \quad \frac{BD}{AB} = \frac{y}{x}. \]

Thus, in triangle $ABD$, $BT$ bisects angle $B$ and $T$ lies on the bisectors of both angle $A$ and $B$. Therefore $T$ is the incentre of triangle $ABC$.

3896. Proposed by Dao Thanh Oai and Nguyen Minh Ha.

Let $[WXYZ]$ represent the signed area of the quadrilateral $WXYZ$ (where $W, X, Y, Z$ can be any four points in the plane), namely half the signed area of the parallelogram formed by the vectors $\vec{WY}$ and $\vec{XZ}$:

\[ [WXYZ] = \frac{1}{2} |\vec{WY}||\vec{XZ}| \sin(\vec{WY}, \vec{XZ}). \]

If $A_1A_2 \ldots A_{2n}$ and $B_1B_2 \ldots B_{2n}$ are two similarly oriented regular $2n$-gons in the plane, prove that $[A_1A_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ is constant for any $i$, $1 \leq i \leq 2n$, where the indices are reduced modulo $2n$.

We received four correct submissions. We present the solution independently submitted by Michel Bataille and by C.R. Pranesachar.

We first show that if the points $W, X, Y, Z$ are represented by the complex numbers $w, x, y, z$, respectively, then

\[ [WXYZ] = \frac{1}{2} \text{Im} ((z - x)(\overline{y} - \overline{w})). \] (1)
Proof. Let $\alpha = \arg(z - x)$ and $\beta = \arg(y - w)$. Then, $\sin(\overrightarrow{WY}, \overrightarrow{XZ}) = \sin(\alpha - \beta)$, whence

$$[WXYZ] = \frac{1}{2} |y - w| \cdot |z - x| \sin(\alpha - \beta) = \frac{1}{2} \left| y - w \right| |z - x| \text{Im}(e^{i(\alpha - \beta)}) = \frac{1}{2} \text{Im}(|z - x| e^{i\alpha} \cdot |y - w| e^{-i\beta});$$

that is, $[WXYZ] = \frac{1}{2} \text{Im} \left( (z - x)(\overline{y - w}) \right)$ as claimed.

Turning to the problem, we will suppose without loss of generality that the affixes of vertices $A_1, A_2, \ldots, A_{2n}$ of the first regular $2n$-gon are

$$a_1 = \omega, a_2 = \omega^2, \ldots, a_{2n} = \omega^{2n} = 1,$$

where $\omega = e^{\pi i/n}$. Because $B_1B_2 \ldots B_{2n}$ is directly similar to $A_1A_2 \ldots A_{2n}$, the affixes of its vertices are

$$b_1 = a \omega + b, \quad b_2 = a \omega^2 + b, \ldots, \quad b_{2n} = a + b$$

for some complex numbers $a, b$ with $a \neq 0$. Then, using (1), we obtain

$$2[A_n A_{n+i} B_{n+i+1} B_{i+1}] = \text{Im}((b_i - a_i+1) \overline{(b_{i+1} - a_{i+1}))}
= \text{Im}(a\omega^i + b - \omega^{i+1}(\overline{a \omega^{-i} + b} - \omega^{-i}))
= \text{Im}(c + a\overline{b}\omega^i - b\omega^{i+1} + \overline{a b} \omega^{-i} + b \omega^{-i}),$$

where $c = \frac{|a|^2}{\omega} + |b|^2 - (a + \overline{\omega}) + \omega$.

Similarly,

$$2 [A_n A_{n+i+1} B_{n+i+1} B_{n+i}] = \text{Im}(c + a\overline{b}\omega^{n+i} - b\omega^{n+i+1} + \overline{a b} \omega^{-(n+i+1)} + b \omega^{-(n+i)}),$$

$$= \text{Im}(c - a\overline{b}\omega^i + b\omega^{i+1} - \overline{a b} \omega^{-(i+1)} + b \omega^{-i}),$$

where the second equality follows from $\omega^n = e^{i\pi} = -1$. Thus,

$$[A_i A_{i+1} B_{i+1} B_i] + [A_{n+i} A_{n+i+1} B_{n+i+1} B_{n+i}] = \frac{1}{2} \text{Im}(2c) = (1 - |a|^2) \sin \frac{\pi}{n}.$$ 

This quantity is independent of $i$ ($1 \leq i \leq 2n$), and the proof is complete.

Editor’s Comments. The solutions of Bataille and Pranesachar were nearly identical except for notation — Pranesachar used Cartesian coordinates instead of Bataille’s complex numbers. The use of complex numbers resulted in a more compact presentation.

3897. Proposed by Yakub Aliyev.

Let the cevians $BB_1$ and $CC_1$ of a triangle $ABC$ intersect at the point $O$. Prove that if a line is drawn through $O$ meeting line segment $BC_1$ at $X$ and line segment $B_1C$ at $Y$, then

$$\frac{|BX|}{|XC_1|} > \frac{|B_1Y|}{|YC|}.$$
We received ten correct submissions. We present the solution by Roy Barbara.

Let $\ell$ denote the line through $C$ that is parallel to $AB$. Denote by $B_2$ and $Z$ the points where $\ell$ meets $BB_1$ and $XY$, respectively. Finally, the line through $B_1$ that is parallel to $YZ$ meets $\ell$ at $U$. Because all points of $\ell$ except $C$ lie outside the given triangle, $B_1$ must lie between $O$ and $B_2$, so that $U$ lies between $Z$ and $B_2$, which implies that

$$B_2Z > UZ.$$  \hfill (1)

Similar triangles $BXO$ and $B_2ZO$ provide $\frac{BX}{B_2Z} = \frac{XO}{ZO}$; similar triangles $C_1XO$ and $CZO$ provide $\frac{XC}{ZC} = \frac{ZO}{CO}$. From this pair of equalities we get

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC}.$$  \hfill (2)

Since $YZ$ is parallel to the base $UB_1$ of $\Delta UB_1C$, we have

$$\frac{B_1Y}{YC} = \frac{UZ}{ZC}.$$  \hfill (3)

Using (2), (1), (3) in this order we obtain

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC} > \frac{UZ}{ZC} = \frac{B_1Y}{YC}.$$  

Editors comments. Only Bataille observed explicitly that it is necessary to assume that the line segments mentioned in the statement of the problem exclude their endpoints. Moreover, everybody assumed tacitly that the points $B_1$ and $C_1$ were chosen to lie on the sides of $\Delta ABC$. Should they lie on the sides extended away from the vertex $A$ beyond $B$ and $C$, then the featured solution could be easily modified to prove that the required inequality would be reversed.

3898. Proposed by Dragoljub Milošević.

On the extension of the side $AB$ of the regular pentagon $ABCDE$, let the points $F$ and $G$ be placed in the order $F,A,B,G$ so that $AG = BF = AC$. Compare the area of triangle $FGD$ to the area of pentagon $ABCDE$.

There were 14 solutions, all of them correct. Seven solvers used variants of the strategy of Solution 1. All but one of the remaining solvers, with more or fewer complications, provided a computational solution along the lines of Solution 3. However, Dag Jonsson had a different slant, which we reproduce in Solution 2.

Solution 1, by various solvers.

Since any side of a regular pentagon is parallel to the diagonal not containing either of its endpoints, $BF \parallel EC$. Since $BF = EC$, $BFEC$ is a parallelogram and $FE \parallel BC \parallel DA$. Therefore $[ADF] = [ADE]$. Also, $[BDG] = [BDC]$. Hence

$$[ABCDE] = [ABD] + [ADE] + [BDC] = [ABD] + [ADF] + [BDG] = [FGD].$$
Solution 2, by Dag Jonsson modified by the editor.

Let \( s \) and \( d \) be the side and diagonal lengths of the pentagon and let \( AC \) and \( BD \) intersect at \( H \). From the similarity of the triangles \( ADB \) and \( BAH \), we deduce that \( s/d = (d-s)/s \), whence \( s^2 - d^2 + ds = 0 \). The pentagon \( ABCDE \) is the union of five \( 72^\circ - 54^\circ - 54^\circ \) isosceles triangles whose bases are the sides of the pentagon of length \( s \) and whose common apex is the circumcentre of the pentagon. Each such triangle has area \( ks^2 \) for some constant \( k \).

Since \( AG = AD \), it follows that

\[ \angle GFD = \angle FGD = \angle AGD = \angle ADG = \frac{1}{2}(180^\circ - \angle DAG) = 54^\circ, \]

so that triangle \( FGD \) is a \( 72^\circ - 54^\circ - 54^\circ \) isosceles triangle with base \( 2d-s \) similar to each component triangle of the pentagon. Hence its area is \( k(2d-s)^2 \) and so

\[ \boxed{[ABCDE] - [FGD] = 5ks^2 - k(2d-s)^2 = 4k(s^2 - d^2 + ds) = 0}. \]

Solution 3, by various solvers.

If \( t = \cos 36^\circ \), then \( -t = \cos 144^\circ = 2(2t^2 - 1)^2 - 1 = 8t^4 - 8t^2 + 1 \). Since \( 0 = 8t^4 - 8t^2 + t + 1 = (t+1)(2t-1)(4t^2 - 2t - 1) \) and \( t \neq -1, \frac{1}{2} \), we have that \( \cos 36^\circ = \frac{1}{4}(1 + \sqrt{5}) \) from which \( \cot 36^\circ = \frac{1}{\sqrt{25 + 10\sqrt{5}}} \). Let \( 1 \) be the side length of the pentagon. Then the diagonal length is \( 2 \cos 36^\circ = \frac{1}{2}(1 + \sqrt{5}) \) and the length of \( FG \) is \( 4 \cos 36^\circ - 1 = \sqrt{5} \).

The distance from the circumcentre to each side of the pentagon is \( \frac{1}{2} \tan 54^\circ \), so

\[ [ABCDE] = \frac{5}{4} \tan 54^\circ = \frac{5}{4} \cot 36^\circ = \frac{\sqrt{25 + 10\sqrt{5}}}{4}. \]

The distance from each vertex of the pentagon to its opposite edge is \( \frac{1}{4} \tan 72^\circ \), so

\[ [FGD] = \frac{\sqrt{5}}{4} \tan 72^\circ = \frac{\sqrt{5}}{2} \left( \frac{\cot 36^\circ}{\cot^2 36^\circ - 1} \right) = \frac{5\sqrt{5}}{2} \left( \frac{\sqrt{25 + 10\sqrt{5}}}{10\sqrt{5}} \right) = \frac{\sqrt{25 + 10\sqrt{5}}}{4} = [ABCDE]. \]

3899. Proposed by George Apostolopoulos.

Let \( a, b \) and \( c \) be positive real numbers such that \( a + b + c = 3 \). Prove that

\[ \left( \frac{a^3 + 1}{a^2 + 1} \right)^2 + \left( \frac{b^3 + 1}{b^2 + 1} \right)^2 + \left( \frac{c^3 + 1}{c^2 + 1} \right)^2 \geq ab + bc + ca. \]

When does equality hold?
We received 20 correct submissions. There were a variety of different solutions to this problem; we will feature two solutions.

Solution 1, by Salem Malikić.

Editor's comment. This approach, utilized by most of the solvers, consists of first finding a lower bound for the square expression on the left hand side and then showing that the right hand side is a lower bound for the resulting function.

By the Cauchy-Schwarz inequality we have that for all positive reals $x$:

$$(x^3 + 1)(x + 1) \geq (x^4 \cdot x^2 + 1)^2 = (x^2 + 1)^2.$$

Using this inequality we obtain

$$\left(\frac{x^3 + 1}{x^2 + 1}\right)^2 = \frac{(x^3 + 1)(x^2 - x + 1)}{(x^2 + 1)^2} \geq \frac{(x^2 + 1)^2(x^2 - x + 1)}{(x^2 + 1)^2} = x^2 - x + 1.$$ 

Then

$$\left(\frac{a^3 + 1}{a^2 + 1}\right)^2 + \left(\frac{b^3 + 1}{b^2 + 1}\right)^2 + \left(\frac{c^3 + 1}{c^2 + 1}\right)^2 \geq (a^2 + b^2 + c^2) - (a + b + c) + 3$$

$$= a^2 + b^2 + c^2$$

$$= \left(\frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} + (ab + bc + ca)\right)$$

$$\geq ab + bc + ca.$$

From the last inequality, it is obvious that equality is only achieved by $a = b = c = 1$.

Solution 2, by Oliver Geupel.

Editor's comment. This is a more involved but also more general solution. The problem is solved with the method of Lagrange multipliers which was recently presented in this journal, see [2013 : 24].

Let us start with the following preliminary observation: The function

$$h : [0, 3] \to \mathbb{R} : x \mapsto \frac{2x(x^3 + 1)(x^3 + 3x - 2)}{(x^2 + 1)^3} + x - 3$$

is convex on $[0, 1]$ and increasing on $[1, 3]$ – as can be seen fairly easily from the first and second derivatives – and satisfies $h(0) < h(1) < h(3)$, implying $h(x) < h(3)$ for all $x \in [0, 1]$. Hence, for real numbers $a, b \in [0, 3]$ with the property $a + b = 3$, the condition $h(a) = h(b)$ can only be satisfied if $a = b = 3/2$. Furthermore, for real numbers $a, b, c \in [0, 3]$ with the property $a + b + c = 3$, the condition $h(a) = h(b) = h(c)$ can only be satisfied if $a = b = c = 1$.

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Now let us turn to the proposed problem. Denote by $m$ the minimum value of the continuous function

$$f(x, y, z) = \left(\frac{x^3 + 1}{x^2 + 1}\right)^2 + \left(\frac{y^3 + 1}{y^2 + 1}\right)^2 + \left(\frac{z^3 + 1}{z^2 + 1}\right)^2 - (xy + yz + zx)$$

on the compact region

$$K = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0 \text{ and } x + y + z = 3\}.$$

We prove that $m = 0$ and that $f(a, b, c) = 0$ is equivalent to $a = b = c = 1$.

Assume that $f(a, b, c) = m$. We have $m \leq f(1, 1, 1) = 0$.

First we show that $(a, b, c)$ is an interior point of $K$. Suppose $(a, b, c)$ is on the boundary of $K$, say $c = 0$. Since $f(3, 0, 0) = f(0, 3, 0) > 0$, the point $(a, b, c)$ cannot be a vertex of the triangular region $K$. Hence, $(a, b)$ is an interior point of the region

$$\{(x, y) \in \mathbb{R}^2 : x, y \geq 0 \text{ and } x + y = 3\}.$$

Then, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$h(a) = \partial_1 f(a, b, 0) = \lambda = \partial_2 f(a, b, 0) = h(b).$$

By our preliminary observation, we obtain $a = b = 3/2$, which is impossible because $f(3/2, 3/2, 0) > 0$. We have proved that $(a, b, c)$ is an interior point of $K$.

Now, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$h(a) = \partial_1 f(a, b, c) = \lambda, \quad h(b) = \partial_2 f(a, b, c) = \lambda, \quad h(c) = \partial_3 f(a, b, c) = \lambda.$$

By the preliminary observation, we obtain $a = b = c = 1$.

As a consequence, the proposed inequality follows. The equality holds if and only if $a = b = c = 1$.

**3900. Proposed by Abdulkadir Altintaş and Halit Çelik.**

In a triangle $ABC$, $AB = AC$, $m(BAC) = 20^\circ$, $D$ is the point on $AC$ such that $m(DBC) = 25^\circ$ and $E$ is the point on $AB$ such that $m(BCE) = 65^\circ$. Find the measure of the angle $CED$.

We received 14 correct solutions, and 12 incorrect or incomplete solutions, most of which used calculators and/or rounded values. We present two solutions.

**Solution 1**, by Dag Jonsson, slightly modified by the editor.

Note that $\angle ABC = \angle ACB = 80^\circ$ ($\triangle ABC$ is isosceles), which allows us to calculate $\angle DCE = 15^\circ$, $\angle DBE = 55^\circ$, $\angle BEC = 35^\circ$ and $\angle BDC = 75^\circ$.

We draw the normal $EF$ to the side $AC$. Let $\alpha = \angle EDF$, then

$$\frac{AF}{FD} = \frac{EF}{FD} \div \frac{EF}{AF} = \frac{\tan \alpha}{\tan 20^\circ}. \quad (1)$$

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The Sine Law applied to $\triangle ABD$ gives
\[
\frac{AD}{AB} = \frac{\sin 55^\circ}{\sin (180^\circ - 75^\circ)} = \frac{\sin 55^\circ}{\sin 75^\circ}, \tag{2}
\]
and similarly from $\triangle AEC$ we find that
\[
\frac{AE}{AC} = \frac{\sin 15^\circ}{\sin 35^\circ}. \tag{3}
\]
Since $AB = AC$, we get from (2) and (3) that
\[
\frac{AE}{AD} = \frac{\sin 15^\circ \cdot \sin 75^\circ}{\sin 35^\circ \cdot \sin 55^\circ}.
\]
By construction, $\triangle AFE$ is a right triangle, and so $AF = AE \cdot \sin 70^\circ$. Hence,
\[
\frac{AF}{AD} = \frac{AE \cdot \sin 70^\circ}{AD} = \frac{\sin 15^\circ \cdot \sin 75^\circ \cdot \sin 70^\circ}{\sin 35^\circ \cdot \cos 15^\circ \cdot \sin 70^\circ} = \frac{\frac{1}{2} \sin 30^\circ \cdot \sin 70^\circ}{\frac{1}{2} \sin 70^\circ} = \frac{1}{2}.
\]
But this means that $AF = FD$, and by (1) that $\tan \alpha = \tan 20^\circ$, i.e. $\alpha = 20^\circ$. Thus, $\angle CED = \alpha - \angle ECD = 5^\circ$.

Solution 2, by C.R. Pranesachar, modified by the editor.

We shall show that $\angle CED = 5^\circ$. Denote by $O$ the intersection of $BD$ and $CE$. Since $\angle BOC = 180^\circ - (\angle OBC + \angle OCB) = 180^\circ - (25^\circ + 65^\circ) = 90^\circ$, $BD$ and $CE$ intersect at right angles. We also calculate that $\angle ABC = \angle ACB = 80^\circ$, and hence $\angle EBD = 55^\circ$, $\angle ECD = 15^\circ$.

Denote by $a$ the length of $BC$. From the right angle triangles around $O$, we get $BO = a \cos 25^\circ$; $EO = BO \tan 55^\circ = a \cos 25^\circ \tan 55^\circ$; $CO = a \sin 25^\circ$; $DO = CO \tan 15^\circ = a \sin 25^\circ \tan 15^\circ$.

Let $\theta = \angle CED$. From $\triangle DOE$ we have
\[
\tan \theta = \frac{DO}{OE} = \frac{a \sin 25^\circ \tan 15^\circ}{a \cos 25^\circ \tan 55^\circ} = \frac{\tan 25^\circ \tan 15^\circ}{\tan 55^\circ}.
\]
For readability, denote $\tan 5^\circ$ by $t$. Then, using the difference of angles formula for tangent and simplifying, we get
\[
\tan \theta = \tan 15^\circ \cdot \frac{\tan(30^\circ - 5^\circ)}{\tan(60^\circ - 5^\circ)} = \tan 15^\circ \cdot \frac{1 - \sqrt{3t}}{\sqrt{3} + t} \cdot \frac{1 + \sqrt{3t}}{\sqrt{3} - t} = \tan 15^\circ \cdot t \cdot \frac{1 - 3t^2}{3t - t^3}.
\]
By the triple angle formula for tangent, $\frac{1 - 3t^2}{3t - t^3} = \frac{1}{\tan(3 \cdot 5^\circ)}$, so it follows from the above calculation that $\tan \theta = t = \tan 5^\circ$. But $\theta \in [0, 180^\circ]$, so it follows that $\theta = 5^\circ$, as claimed.