SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2013: 39(10), p. 456–460.

3891. *Proposed by Michel Bataille.*

Let ABC be a triangle that is neither equilateral nor right-angled and let O be its circumcentre. Let the altitudes from A, B and C meet the circumcircle again at A', B' and C', respectively. If U, V and W denote the circumcentres of triangles OBC, OCA and OAB, respectively, prove that the lines UA', VB' and WC' are concurrent and identify their common point.

We received seven solutions, all correct. We present four of these solutions.

Solution 1, by the proposer.

Let $\mathbf{I}$ be the inversion in the circumcircle Γ of triangle ABC and $\mathbf{J}$ be the inversion in the circumcircle Δ of triangle OBC. Let H be the orthocentre of triangle ABC. The points H and A' are reflections of each other in the axis BC; any circle through these two points is orthogonal to BC. The inversion $\mathbf{I}$ fixes A' and carries the line BC to Δ. Let $P = \mathbf{I}(H)$. Then, because $\mathbf{I}$ preserves orthogonality, any circle through A' and P is orthogonal to Δ, so that $\mathbf{J}(A') = P$. Therefore $A'P$ passes through the centre U of Δ. [See R.A. Johnson, *Advanced Euclidean Geometry*, Houghton-Mifflin, 1929, reprinted by Dover, 1960, paragraph 80, p. 55.] Thus UA', and similarly VB' and WC', all pass through the point $P = \mathbf{I}(H)$.

Solution 2, by Oliver Geupel (slightly modified by the editor).

We prove that the three lines are concurrent at the inverse point P of the orthocentre H with respect to inversion in the circumcircle of the triangle ABC.

Consider the problem in the complex plane where the circumcircle is the unit circle and the geometric inverse of z is $1/\bar{z}$. Let a, a', b, c, h, p, u denote the affixes of the respective points A, A', B, C, H, P, U. Noting that two vectors z and w are perpendicular if and only if $z/\bar{z} = -w/\bar{w}$, we deduce from $AA' \perp BC$ that

$$-aa' = \frac{a-a'}{(1/a)-(1/a')} = \frac{a-a'}{\bar{a}-\bar{a'}} \quad \text{and} \quad -\frac{b-c}{\bar{b}-\bar{c}} = -\frac{b-c}{(1/b)-(1/c)} = bc,$$

whence

$$a' = -\frac{bc}{a} \qquad \text{and} \qquad \bar{a'} = -\frac{a}{bc}.$$

The condition $u\bar{u} = (u-b)(\bar{u}-\bar{b})$ leads to $\bar{b}u + b\bar{u} = 1$. Similarly $\bar{c}u + c\bar{u} = 1$. From this, we find that

$$u = \frac{bc}{b+c} \qquad \text{and} \qquad \bar{u} = \frac{1}{b+c}.$$

Since $h = a + b + c$ and $h\bar{p} = 1$,

$$\bar{p} = \frac{1}{a+b+c} \qquad \text{and} \qquad p = \frac{abc}{ab+bc+ca}.$$

A straightforward computation yields

$$\begin{aligned}(a' - u)(\bar{a'} - \bar{p}) &= \left(\frac{bc}{a} + \frac{bc}{b+c}\right)\left(\frac{a}{bc} + \frac{1}{a+b+c}\right) = \frac{a^2+ab+bc+ca}{a(b+c)} \\ &= \left(\frac{a}{bc} + \frac{1}{b+c}\right)\left(\frac{bc}{a} + \frac{abc}{ab+bc+ca}\right) = (\bar{a'} - \bar{u})(a' - p).\end{aligned}$$

Hence the complex number $(a' - u)/(a' - p)$, being equal to its conjugate, is real. Thus the line $A'U$ passes through the point P. Similarly, the lines $B'V$ and $C'W$ pass through P. This completes the proof.

Solution 3, by Titu Zvonaru.

Conventionally, let a, b, c, R and H be the sides, circumradius and orthocentre of triangle ABC. Suppose D on BC is the foot of the altitude from A. It is known that $HD = DA' = 2R\cos B\cos C$. Since BC subtends the angle $2A$ at O and the angle $180^\circ - 2A$ at any point on the arc of the circumcircle of OBC opposite O, it follows that

$$OU = BU = \frac{a}{2\sin 2A} = \frac{2R\sin A}{4\sin A\cos A} = \frac{R}{2\cos A}.$$

Let UA' and OH intersect at P. Because $OU \| HA'$,

$$\frac{PO}{PH} = \frac{OU}{HA'} = \frac{R}{2\cos A} \cdot \frac{1}{4R\cos B\cos C} = \frac{1}{8\cos A\cos B\cos C}.$$

Therefore, UA', VB' and WC' are concurrent at P, which lies on the Euler line OH and satisfies

$$\frac{PO}{PH} = \frac{1}{8\cos A\cos B\cos C}.$$

(This argument can be adapted if one angle is obtuse.)

Solution 4, by Prithwijit De.

We assume that the triangle is acute and not equilateral. Let $X = UW \cap OB$ and $Y = UV \cap OC$. Since UW right bisects OB, $\angle UXO = 90^\circ$. Similarly, $\angle UYO = 90^\circ$, so that the quadrilateral $UXOY$ is concyclic. Therefore,

$$\angle WUV = \angle XUY = 180^\circ - \angle XOY = 180^\circ - \angle BOC = 180^\circ - 2A = \angle C'A'B'.$$

To see the last equality, note that

$$\angle AA'C' = \angle ACC' = 90^\circ - \angle BAC = \angle ABB' = \angle AA'B$$

(so that the line AA' bisects $\angle B'A'C'$). Similarly, $\angle VWU = \angle B'C'A'$ and $\angle UVW = \angle A'B'C'$. Thus triangles $A'B'C'$ and UVW are directly similar, and the lines UA', VB' and WC' concur at the centre P of homothety.

Since the line AA' bisects $\angle B'A'C'$, and, similarly, BB' and CC' bisects respectively $\angle A'B'C'$ and $\angle A'C'B'$, the incentre of triangle $A'B'C'$ is the orthocentre H of triangle ABC.

Since O is the midpoint of the arc BC of the circumcircle with centre U of triangle OBC, $\angle OUB = \angle OUC$, and OU bisects $\angle WUV$. Similarly OV bisects $\angle UVW$ and OW bisects $\angle UWV$. Therefore the incentre of triangle UVW is O. The line joining the incentres of triangles $A'B'C'$ and UVW passes through the centre of homothety, so that O, H and P are collinear.

From solution 2, note that $A'B'/UV = A'H/UO = 8\cos A\cos B\cos C$. Also

$$\begin{aligned} OH^2 &= (a/2 - c\cos B)^2 + (2R\cos B\cos C - R\cos A)^2 \\ &= R^2[(\sin A - 2\sin C\cos B)^2 + (2\cos B\cos C - \cos A)^2] \\ &= R^2[1 - 4\sin A\sin C\cos B + 4\cos^2 B - 4\cos A\cos B\cos C] \\ &= R^2[1 - 4\cos B(\cos A\cos C + \sin A\sin C + \cos(A+C)] \\ &= R^2[1 - 8\cos A\cos B\cos C]. \end{aligned}$$

Therefore,

$$1 - \left(\frac{OH}{R}\right)^2 = 8\cos A\cos B\cos C = \frac{A'B'}{UV} = \frac{PH}{PO}.$$

This simplifies to $OH \cdot OP = R^2$, from which we see that P is the inversion of H in the circumcircle of triangle ABC.

3892. *Proposed by George Apostolopoulos.*

Let a, b and c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that

$$\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3 + b^3 + c^3}{3abc}.$$

We received 11 correct solutions and one incorrect solution. We present a composite of similar solutions by Šefket Arslanagić, Michel Bataille, Scott Brown, John Heuver, Kee-Wai Lau, Dragoljub Milošević, and Titu Zvonaru.

Using the identities

$$\cos A + \cos B + \cos C = \frac{R+r}{R},$$

$$a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$$

and

$$abc = 4Rrs,$$

[for example in D. S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989], the claimed inequality is equivalent to

$$\frac{R}{r} \geq \frac{2}{3}\left(\frac{R+r}{R}\right) + \frac{2s(s^2-3r^2-6Rr)}{12Rrs}.$$

This can be rewritten as $6R^2+2Rr-r^2 \geq s^2$. Gerretsen's Inequality asserts that

$$4R^2+4Rr+3r^2 \geq s^2,$$

so it suffices to show that

$$6R^2+2Rr-r^2 \geq 4R^2+4Rr+3r^2.$$

But this is equivalent to

$$0 \leq 2R^2-2Rr-4r^2 = 2(R-2r)(R+r),$$

which holds by Euler's inequality $R \geq 2r$.

3893. *Proposed by Ovidiu Furdui.*

Let $n \geq 1$ be an integer and let the decimal part of a real number a be defined by $\{a\} = a - \lfloor a \rfloor$. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin 2x \left\{\ln^{2n-1} \tan x\right\} \mathrm{d}x.$$

We received six correct solutions. We present the solution of the AN-anduud Problem Solving Group.

With the substitution $u = \tan x$, the integral becomes

$$\begin{aligned} I &= \int_0^{\infty} \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du \\ &= \int_0^{1} \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du + \int_1^{\infty} \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du. \end{aligned}$$

If in this first integral we substitute $v = \frac{1}{u}$, we obtain

$$\begin{aligned} I &= \int_1^{\infty} \frac{2v}{(1+v^2)^2}\{-\log^{2n-1} v\}\, dv + \int_1^{\infty} \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du \\ &= \int_1^{\infty} \frac{2x}{(1+x^2)^2}\left(\{-\log^{2n-1} x\} + \{\log^{2n-1} x\}\right) dx \\ &= \int_1^{\infty} \frac{2x}{(1+x^2)^2}\, dx = \frac{1}{2}, \end{aligned}$$

using the equation $\{y\} + \{-y\} = 1$, which holds for all $y \in \mathbb{R} \setminus \mathbb{Z}$.

3894. *Proposed by Paul Bracken.*

a) Prove that for $n \in \mathbb{N}$,

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

b) For the equation in part a), evaluate the limit of the right-hand side as $n \to \infty$ and compute the sum in closed form.

We received 12 correct submissions. We present two solutions for part a, and three solutions for part b, each utilized by multiple solvers.

For ease of solution-wide notation, let S_n be the left-hand-side of the identity, and let T_n be the right-hand-side.

Part a, Solution 1. It is easily checked that $\frac{1}{(3k)^3-3k} = \frac{1}{2}\left(\frac{1}{3k-1} + \frac{1}{3k+1} - \frac{2}{3k}\right)$. It follows that

$$\begin{aligned} 1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} &= \sum_{k=0}^{n} \frac{1}{3k+1} + \sum_{k=1}^{n} \frac{1}{3k-1} - 2\sum_{k=1}^{n} \frac{1}{3k} \\ &= \left(\sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{3k}\right) - 2\sum_{k=1}^{n} \frac{1}{3k} \\ &= \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{3n+1} \frac{1}{k} \end{aligned}$$

that is,

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

Part a, Solution 2. It is simple to check that for $n = 1$, both sides of the proposed equality evaluate to $\frac{13}{12}$. After assuming that the identity is true for n, in order to prove that it is also true for $n+1$ it is enough to prove that $S_{n+1} - S_n = T_{n+1} - T_n$, for then we may add $S_n = T_n$ to both sides to obtain the result. So we must prove

$$\frac{2}{(3(n+1))^3 - 3(n+1)} = \sum_{k=1}^{2n+3} \frac{1}{k+n+1} - \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

But the right-hand side of the last equation is

$$\frac{1}{3n+4} + \frac{1}{3n+3} + \frac{1}{3n+2} - \frac{1}{n+1} = \frac{2}{(3(n+1))^3 - 3(n+1)}.$$

Part b, Solution 1. From the above calculation in part a, the sum $T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n}$ satisfies

$$T_n = \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = H_{3n+1} - H_n$$

where as usual H_m denotes $\sum_{k=1}^{m} \frac{1}{k}$. It is well-known that for all positive integer m, we have $H_m = \ln(m) + \gamma + \varepsilon(m)$ where γ is the Euler-Mascheroni constant and $\lim_{m\to\infty} \varepsilon(m) = 0$. As a result, we may write

$$T_n = \ln(3n+1) + \gamma + \varepsilon(3n+1) - \ln(n) - \gamma - \varepsilon(n) = \ln\left(\frac{3n+1}{n}\right) + \varepsilon(3n+1) - \varepsilon(n)$$

and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\left(3 + \frac{1}{n}\right) = \ln(3).$$

Part b, Solution 2. The right-hand side of the identity is bounded:

$$\int_{n+1}^{3n+2} \frac{1}{x}\,dx \le \sum_{k=n+1}^{3n+1} \frac{1}{k} \le \int_{n}^{3n+1} \frac{1}{x}\,dx.$$

Evaluating both sides of the inequality gives

$$\ln\left(\frac{3n+2}{n+1}\right) \le \sum_{k=n+1}^{3n+1} \frac{1}{k} \le \ln\left(\frac{3n+1}{n}\right),$$

and taking the limit as n tends to infinity yields

$$1 + 2\sum_{k=1}^{\infty} \frac{1}{(3k)^3 - 3k} = \lim_{n\to\infty} T_n = \ln(3).$$

Part b, Solution 3. Observe that we have:

$$\begin{aligned} T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n} &= \sum_{k=1} n\frac{1}{k+n} + \sum_{k=n+1}^{2n} \frac{1}{k+n} + \frac{1}{3n+1} \\ &= \sum_{k=1} n\frac{1}{1+\frac{k}{n}}\frac{1}{n} + \sum_{k=1} n\frac{1}{2+\frac{k}{n}}\frac{1}{n} + \frac{1}{3n+1}. \end{aligned}$$

The two sums in the last line are Riemann sums for integrals of $\frac{1}{1+x}$ and $\frac{1}{2+x}$ over $[0,1]$, respectively, with constant step size equal to $\frac{1}{n}$, and right-hand endpoints. Taking n to infinity gives:

$$\begin{aligned} \lim_{n\to\infty} T_n &= \int_0^1 \frac{1}{1+x}\,dx + \int_0^1 \frac{1}{2+x}\,dx + \lim_{n\to\infty} \frac{1}{3n+1} \\ &= \ln(2) - \ln(1) + \ln(3) - \ln(2) + 0 = \ln(3). \end{aligned}$$

Editor's Comments. All solvers basically used some combination of these solutions. The induction proof in Part 1, Solution 2 is a difference equation technique. The Riemann sum argument is clever, but needs careful treating; that is, T_n itself is *not* a Riemann sum, but it *is* a Riemann sum plus a term that goes to 0. Finally, the comment about "a closed form" was, I believe, intended to mean "compute the value of the limit of this expression", not to find a closed form expression for the finite sum T_n.

3895. *Proposed by Neculai Stanciu and Titu Zvonaru.*

In the acute triangle ABC with $AB \neq AC$, let A' be the foot of the altitude from A, and let the bisector of the angle at A meet BC at D and the circumcircle at M. Finally, for a point T on the segment AD, let P and N be its projections on AA' and BC, respectively. Prove that if M, N, and P are collinear, then T is the incentre of the triangle.

Seven solutions were submitted, all correct. We give two solutions.

Solution 1, by Michel Bataille.

We first establish that T is the incentre of triangle ABC if and only if $MB = MT$. For, T is the incentre $\Leftrightarrow$ it lies on the bisector of angle $B \Leftrightarrow \angle CBT = \angle ABT$

$$\Leftrightarrow \angle TBM - \angle CBM = \angle BTM - \angle BAM = \angle BTM - \angle CBM$$
$$\Leftrightarrow \angle TBM = \angle BTM \Leftrightarrow MB = MT.$$

Suppose that M, N, P are collinear and that U is the midpoint of BC. Wolog, let $b = AC > AB = c$ so that $DC > DB$ (since $DC : DB = b : c$). Since $BU/BM = \cos(\angle MBC) = \cos(\angle MAC) = \cos(A/2)$, it follows that $MB = a/(2\cos A/2)$, where $a = BC$. Let M' be the point diametrically opposite to M on the circumcircle of triangle ABC. Since $\angle MAM' = 90^\circ$ and

$$\angle AMM' = 90^\circ - \angle AM'M = 90^\circ - \angle ACM = 90^\circ - \left(C + \frac{A}{2}\right) = \frac{B-C}{2},$$

we have that

$$MA = MM' \cos(\angle AMM') = 2R\cos\frac{B-C}{2}.$$

Also

$$\frac{a}{2}\tan\frac{A}{2} = BU\tan\frac{A}{2} = MU = MD\cos\frac{B-C}{2},$$

whereupon

$$MD = \frac{a\tan A/2}{2\cos((B-C)/2).}$$

Since $PT \| ND$ and $AP \| NT$, the homothety with centre M that takes $N \longrightarrow P$ also takes $D \longrightarrow T$ and $T \longrightarrow A$. Therefore $MA : MT = MT : MD$, and so

$$MT^2 = MA \cdot MD = 2R\cos\frac{B-C}{2} \cdot \frac{a\tan A/2}{2\cos((B-C)/2)}$$
$$= aR\tan\frac{A}{2} = \frac{a^2}{2\sin A} \cdot \tan\frac{A}{2} = \frac{a^2}{4\cos^2(A/2)} = MB^2.$$

Thus $MT = MB$ and the desired result follows.

Solution 2, by Madhav R. Modak.

Let $AB < AC$, so that the points on BC are in the order $BA'NDC$. Let $AT = x$ and $TD = y$. Since M, N and P are collinear, Menelaus' theorem for triangle $AA'D$ and transversal PNM yields

$$\frac{A'N}{ND} \cdot \frac{DM}{MA} \cdot \frac{AP}{PA'} = -1. \tag{1}$$

Since $AA' \| TN$ and $PT \| A'D$, we have that $AP/PA' = A'N/ND = x/y$, so (1) gives numerically,

$$\frac{DM}{MA} = \frac{y^2}{x^2}. \tag{2}$$

Since $\angle DBM = \angle CAM = \frac{1}{2}A$ and $\angle BMD = \angle BCA = C$, the Sine Law applied to triangle BDM give $BD/\sin C = DM/\sin\frac{1}{2}A$. Since $\angle ABM = B + \frac{1}{2}A$ and $\angle BMA = C$, the Sine Law applied to triangle ABM gives $MA/\sin(B + \frac{1}{2}A) = AB/\sin C$. These give

$$\frac{DM}{MA} = \frac{BD}{AB} \cdot \frac{\sin\frac{1}{2}A}{\sin(B + \frac{1}{2}A)}. \tag{3}$$

In triangle ABD, $\angle ABD = 180^\circ - (B + \frac{1}{2}A)$ and $\angle BAD = \frac{1}{2}A$, so that, by the Sine Law, $BD/\sin\frac{1}{2}A = AB/\sin(B + \frac{1}{2}A)$. With (2) and (3), this yields that

$$\frac{y^2}{x^2} = \frac{DM}{MA} = \left(\frac{BD}{AB}\right)^2 \qquad \text{or} \qquad \frac{BD}{AB} = \frac{y}{x}.$$

Thus, in triangle ABD, BT bisects angle B and T lies on the bisectors of both angle A and B. Therefore T is the incentre of triangle ABC.

3896. *Proposed by Dao Thanh Oai and Nguyen Minh Ha.*

Let $[WXYZ]$ represent the signed area of the quadrilateral $WXYZ$ (where W, X, Y, Z can be any four points in the plane), namely half the signed area of the parallelogram formed by the vectors $\overrightarrow{WY}$ and $\overrightarrow{XZ}$:

$$[WXYZ] = \frac{1}{2}|\overrightarrow{WY}||\overrightarrow{XZ}|\sin(\overrightarrow{WY}, \overrightarrow{XZ}).$$

If $A_1A_2\ldots A_{2n}$ and $B_1B_2\ldots B_{2n}$ are two similarly oriented regular $2n$-gons in the plane, prove that $[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ is constant for any i, $1 \le i \le 2n$, where the indices are reduced modulo $2n$.

We received four correct submissions. We present the solution independently submitted by Michel Bataille and by C.R. Pranesachar.

We first show that if the points W, X, Y, Z are represented by the complex numbers w, x, y, z, respectively, then

$$[WXYZ] = \frac{1}{2}\operatorname{Im}\left((z - x)(\overline{y} - \overline{w})\right). \tag{1}$$

Proof. Let $\alpha = \arg(z-x)$ and $\beta = \arg(y-w)$. Then, $\sin(\overrightarrow{WY}, \overrightarrow{XZ}) = \sin(\alpha - \beta)$, whence

$$\begin{aligned}
[WXYZ] &= \frac{1}{2}\,|y-w|\cdot|z-x|\sin(\alpha-\beta)\\
&= \frac{1}{2}\,|y-w|\cdot|z-x|\mathrm{Im}(e^{i(\alpha-\beta)}) = \frac{1}{2}\,\mathrm{Im}(|z-x|e^{i\alpha}\cdot|y-w|e^{-i\beta});
\end{aligned}$$

that is, $[WXYZ] = \frac{1}{2}\,\mathrm{Im}\left((z-x)(\overline{y-w})\right)$ as claimed.

Turning to the problem, we will suppose without loss of generality that the affixes of vertices $A_1, A_2, \ldots, A_{2n}$ of the first regular $2n$-gon are

$$a_1 = \omega, a_2 = \omega^2, \ldots, a_{2n} = \omega^{2n} = 1,$$

where $\omega = e^{\pi i/n}$. Because $B_1B_2\ldots B_{2n}$ is directly similar to $A_1A_2\ldots A_{2n}$, the affixes of its vertices are

$$b_1 = a\omega + b, \quad b_2 = a\omega^2 + b, \ldots, \quad b_{2n} = a + b$$

for some complex numbers a, b with $a \neq 0$. Then, using (1), we obtain

$$\begin{aligned}
2[A_iA_{i+1}B_{i+1}B_i] &= \mathrm{Im}((b_i - a_{i+1})(\overline{b_{i+1}} - \overline{a_i})\\
&= \mathrm{Im}(a\omega^i + b - \omega^{i+1})(\overline{a}\omega^{-(i+1)} + \overline{b} - \omega^{-i}))\\
&= \mathrm{Im}(c + a\overline{b}\omega^i - \overline{b}\omega^{i+1} + \overline{a}b\omega^{-(i+1)} - b\omega^{-i}),
\end{aligned}$$

where $c = \frac{|a|^2}{\omega} + |b|^2 - (a + \overline{a}) + \omega$.

Similarly,

$$\begin{aligned}
2\quad & [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]\\
&= \mathrm{Im}(c + a\overline{b}\omega^{n+i} - \overline{b}\omega^{n+i+1} + \overline{a}b\omega^{-(n+i+1)} - b\omega^{-(n+i)})\\
&= \mathrm{Im}(c - a\overline{b}\omega^i + \overline{b}\omega^{i+1} - \overline{a}b\omega^{-(i+1)} + b\omega^{-i}),
\end{aligned}$$

where the second equality follows from $\omega^n = e^{i\pi} = -1$. Thus,

$$[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}] = \frac{1}{2}\mathrm{Im}(2c) = (1 - |a|^2)\sin\frac{\pi}{n}.$$

This quantity is independent of i ($1 \leq i \leq 2n$), and the proof is complete.

Editor's Comments. The solutions of Bataille and Pranesachar were nearly identical except for notation — Pranesachar used Cartesian coordinates instead of Bataille's complex numbers. The use of complex numbers resulted in a more compact presentation.

3897. *Proposed by Yakub Aliyev.*

Let the cevians BB_1 and CC_1 of a triangle ABC intersect at the point O. Prove that if a line is drawn through O meeting line segment BC_1 at X and line segment B_1C at Y, then

$$\frac{|BX|}{|XC_1|} > \frac{|B_1Y|}{|YC|}.$$

We received ten correct submissions. We present the solution by Roy Barbara.

Let ℓ denote the line through C that is parallel to AB. Denote by B_2 and Z the points where ℓ meets BB_1 and XY, respectively. Finally, the line through B_1 that is parallel to YZ meets ℓ at U. Because all points of ℓ except C lie outside the given triangle, B_1 must lie between O and B_2, so that U lies between Z and B_2, which implies that

$$B_2Z > UZ. \tag{1}$$

Similar triangles BXO and B_2ZO provide $\frac{BX}{B_2Z} = \frac{XO}{ZO}$; similar triangles C_1XO and CZO provide $\frac{XC_1}{ZC} = \frac{XO}{ZO}$. From this pair of equalities we get

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC}. \tag{2}$$

Since YZ is parallel to the base UB_1 of ΔUB_1C, we have

$$\frac{B_1Y}{YC} = \frac{UZ}{ZC}. \tag{3}$$

Using (2), (1), (3) in this order we obtain

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC} > \frac{UZ}{ZC} = \frac{B_1Y}{YC}.$$

Editors comments. Only Bataille observed explicitly that it is necessary to assume that the line segments mentioned in the statement of the problem exclude their endpoints. Moreover, everybody assumed tacitly that the points B_1 and C_1 were chosen to lie on the sides of ΔABC. Should they lie on the sides extended away from the vertex A beyond B and C, then the featured solution could be easily modified to prove that the required inequality would be reversed.

3898. *Proposed by Dragoljub Milošević.*

On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.

There were 14 solutions, all of them correct. Seven solvers used variants of the strategy of Solution 1. All but one of the remaining solvers, with more or fewer complications, provided a computational solution along the lines of Solution 3. However, Dag Jonsson had a different slant, which we reproduce in Solution 2.

Solution 1, by various solvers.

Since any side of a regular pentagon is parallel to the diagonal not containing either of its endpoints, $BF \| EC$. Since $BF = EC$, $BFEC$ is a parallelogram and $FE \| BC \| DA$. Therefore $[ADF] = [ADE]$. Also, $[BDG] = [BDC]$. Hence

$$[ABCDE] = [ABD] + [ADE] + [BDC] = [ABD] + [ADF] + [BDG] = [FGD].$$

Solution 2, by Dag Jonsson modified by the editor.

Let s and d be the side and diagonal lengths of the pentagon and let AC and BD intersect at H. From the similarity of the triangles ADB and BAH, we deduce that $s/d = (d-s)/s$, whence $s^2 - d^2 + ds = 0$. The pentagon $ABCDE$ is the union of five $72° - 54° - 54°$ isosceles triangles whose bases are the sides of the pentagon of length s and whose common apex is the circumcentre of the pentagon. Each such triangle has area ks^2 for some contant k.

Since $AG = AD$, it follows that

$$\angle GFD = \angle FGD = \angle AGD = \angle ADG = \frac{1}{2}(180° - \angle DAG) = 54°,$$

so that triangle FGD is a $72° - 54° - 54°$ isosceles triangle with base $2d - s$ similar to each component triangle of the pentagon. Hence its area is $k(2d-s)^2$ and so

$$[ABCDE] - [FGD] = 5ks^2 - k(2d-s)^2 = 4k(s^2 - d^2 + ds) = 0.$$

Solution 3, by various solvers.

If $t = \cos 36°$, then $-t = \cos 144° = 2(2t^2-1)^2 - 1 = 8t^4 - 8t^2 + 1$. Since $0 = 8t^4 - 8t^2 + t + 1 = (t+1)(2t-1)(4t^2-2t-1)$ and $t \neq -1, \frac{1}{2}$, we have that $\cos 36° = \frac{1}{4}(1+\sqrt{5})$ from which $\cot 36° = \frac{1}{5}\sqrt{25+10\sqrt{5}}$. Let 1 be the side length of the pentagon. Then the diagonal length is $2\cos 36° = \frac{1}{2}(1+\sqrt{5})$ and the length of FG is $4\cos 36° - 1 = \sqrt{5}$.

The distance from the circumcentre to each side of the pentagon is $\frac{1}{2}\tan 54°$, so

$$[ABCDE] = \frac{5}{4}\tan 54° = \frac{5}{4}\cot 36° = \frac{\sqrt{25+10\sqrt{5}}}{4}.$$

The distance from each vertex of the pentagon to its opposite edge is $\frac{1}{2}\tan 72°$, so

$$\begin{aligned}[FGD] = \frac{\sqrt{5}}{4}\tan 72° &= \frac{\sqrt{5}}{2}\left(\frac{\cot 36°}{\cot^2 36° - 1}\right)\\ &= \frac{5\sqrt{5}}{2}\left(\frac{\sqrt{25+10\sqrt{5}}}{10\sqrt{5}}\right)\\ &= \frac{\sqrt{25+10\sqrt{5}}}{4} = [ABCDE].\end{aligned}$$

3899. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=3$. Prove that

$$\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \geq ab+bc+ca.$$

When does equality hold?

We received 20 correct submissions. There were a variety of different solutions to this problem; we will feature two solutions.

Solution 1, by Salem Malikić.

Editor's comment. This approach, utilized by most of the solvers, consists of first finding a lower bound for the square expression on the left hand side and then showing that the right hand side is a lower bound for the resulting function.

By the Cauchy-Schwarz inequality we have that for all positive reals x:

$$(x^3+1)(x+1) \geq (x^{\frac{3}{2}} \cdot x^{\frac{1}{2}} + 1)^2 = (x^2+1)^2.$$

Using this inequality we obtain

$$\left(\frac{x^3+1}{x^2+1}\right)^2 = \frac{(x^3+1)(x+1)(x^2-x+1)}{(x^2+1)^2} \geq \frac{(x^2+1)^2(x^2-x+1)}{(x^2+1)^2} = x^2-x+1.$$

Then

$$\begin{aligned}
&\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \\
&\qquad \geq (a^2+b^2+c^2) - (a+b+c) + 3 \\
&\qquad = a^2+b^2+c^2 \\
&\qquad = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + (ab+bc+ca) \\
&\qquad \geq ab+bc+ca.
\end{aligned}$$

From the last inequality, it is obvious that equality is only achieved by $a=b=c=1$.

Solution 2, by Oliver Geupel.

Editor's comment. This is a more involved but also more general solution. The problem is solved with the method of Lagrange multipliers which was recently presented in this journal, see [2013 : 24].

Let us start with the following preliminary observation: The function

$$h : [0,3] \to \mathbb{R} : x \mapsto \frac{2x(x^3+1)(x^3+3x-2)}{(x^2+1)^3} + x - 3$$

is convex on $[0,1)$ and increasing on $[1,3]$ – as can be seen fairly easily from the first and second derivatives – and satisfies $h(0) < h(1) < h(3)$, implying $h(x) < h(1)$ for all $x \in [0,1]$. Hence, for real numbers $a, b \in [0,3]$ with the property $a+b=3$, the condition $h(a) = h(b)$ can only be satisfied if $a = b = 3/2$. Furthermore, for real numbers $a, b, c \in [0,3]$ with the property $a+b+c=3$, the condition $h(a) = h(b) = h(c)$ can only be satisfied if $a = b = c = 1$.

Now let us turn to the proposed problem. Denote by m the minimum value of the continuous function

$$f(x,y,z)=\left(\frac{x^3+1}{x^2+1}\right)^2+\left(\frac{y^3+1}{y^2+1}\right)^2+\left(\frac{z^3+1}{z^2+1}\right)^2-(xy+yz+zx)$$

on the compact region

$$K=\{(x,y,z)\in\mathbb{R}^3\ :\ x,y,z\geq 0 \text{ and } x+y+z=3\}.$$

We prove that $m=0$ and that $f(a,b,c)=0$ is equivalent to $a=b=c=1$.

Assume that $f(a,b,c)=m$. We have $m\leq f(1,1,1)=0$.

First we show that (a,b,c) is an interior point of K. Suppose (a,b,c) is on the boundary of K, say $c=0$. Since $f(3,0,0)=f(0,3,0)>0$, the point (a,b,c) cannot be a vertex of the triangular region K. Hence, (a,b) is an interior point of the region

$$\{(x,y)\in\mathbb{R}^2\ :\ x,y\geq 0 \text{ and } x+y=3\}.$$

Then, there is a Lagrange multiplier $\lambda\in\mathbb{R}$ such that

$$h(a)=\partial_1 f(a,b,0)=\lambda=\partial_2 f(a,b,0)=h(b).$$

By our preliminary observation, we obtain $a=b=3/2$, which is impossible because $f(3/2,3/2,0)>0$. We have proved that (a,b,c) is an interior point of K.

Now, there is a Lagrange multiplier $\lambda\in\mathbb{R}$ such that

$$h(a)=\partial_1 f(a,b,c)=\lambda,\quad h(b)=\partial_2 f(a,b,c)=\lambda,\quad h(c)=\partial_3 f(a,b,c)=\lambda.$$

By the preliminary observation, we obtain $a=b=c=1$.

As a consequence, the proposed inequality follows. The equality holds if and only if $a=b=c=1$.

3900. *Proposed by Abdilkadir Altintaş and Halit Çelik.*

In a triangle ABC, $AB=AC$, $m(BAC)=20°$, D is the point on AC such that $m(DBC)=25°$ and E is the point on AB such that $m(BCE)=65°$. Find the measure of the angle CED.

We received 14 correct solutions, and 12 incorrect or incomplete solutions, most of which used calculators and/or rounded values. We present two solutions.

Solution 1, by Dag Jonsson, slightly modified by the editor.

Note that $\measuredangle ABC=\measuredangle ACB=80°$ ($\triangle ABC$ is isosceles), which allows us to calculate $\measuredangle DCE=15°$, $\measuredangle DBE=55°$, $\measuredangle BEC=35°$ and $\measuredangle BDC=75°$.

We draw the normal EF to the side AC. Let $\alpha=\measuredangle EDF$, then

$$\frac{AF}{FD}=\frac{EF}{FD}\div\frac{EF}{AF}=\frac{\tan\alpha}{\tan 20°}. \tag{1}$$

The Sine Law applied to $\triangle ABD$ gives

$$\frac{AD}{AB} = \frac{\sin\,55^\circ}{\sin\,(180^\circ - 75^\circ)} = \frac{\sin\,55^\circ}{\sin\,75^\circ}, \tag{2}$$

and similarly from $\triangle AEC$ we find that

$$\frac{AE}{AC} = \frac{\sin\,15^\circ}{\sin\,35^\circ}. \tag{3}$$

Since $AB = AC$, we get from (2) and (3) that

$$\frac{AE}{AD} = \frac{\sin\,15^\circ \cdot \sin\,75^\circ}{\sin\,35^\circ \cdot \sin\,55^\circ}$$

By construction, $\triangle AFE$ is a right triangle, and so $AF = AE \cdot \sin\,70^\circ$. Hence,

$$\begin{aligned}\frac{AF}{AD} = \frac{AE \cdot \sin\,70^\circ}{AD} &= \frac{\sin\,15^\circ \cdot \sin\,75^\circ \cdot \sin\,70^\circ}{\sin\,35^\circ \cdot \sin\,55^\circ} \\ &= \frac{\sin\,15^\circ \cdot \cos\,15^\circ \cdot \sin\,70^\circ}{\sin\,35^\circ \cdot \cos\,35^\circ} \\ &= \frac{\frac{1}{2}\sin\,30^\circ \cdot \sin\,70^\circ}{\frac{1}{2}\sin\,70^\circ} = \frac{1}{2}.\end{aligned}$$

But this means that $AF = FD$, and by (1) that $\tan\alpha = \tan 20^\circ$, i.e. $\alpha = 20^\circ$. Thus, $\measuredangle CED = \alpha - \measuredangle ECD = 5^\circ$.

Solution 2, by C.R. Pranesachar, modified by the editor.

We shall show that $\measuredangle CED = 5^\circ$. Denote by O the intersection of BD and CE. Since $\measuredangle BOC = 180^\circ - (\measuredangle OBC + \measuredangle OCB) = 180^\circ - (25^\circ + 65^\circ) = 90^\circ$, BD and CE intersect at right angles. We also calculate that $\measuredangle ABC = \measuredangle ACB = 80^\circ$, and hence $\measuredangle EBD = 55^\circ$, $\measuredangle ECD = 15^\circ$.

Denote by a the length of BC. From the right angle triangles around O, we get $BO = a\cos 25^\circ$; $EO = BO\tan 55^\circ = a\cos 25^\circ\tan 55^\circ$; $CO = a\sin 25^\circ$; $DO = CO\tan 15^\circ = a\sin 25^\circ\tan 15^\circ$.
Let $\theta = \measuredangle CED$. From $\triangle DOE$ we have

$$\tan\theta = \frac{DO}{OE} = \frac{a\sin 25^\circ\tan 15^\circ}{a\cos 25^\circ\tan 55^\circ} = \frac{\tan 25^\circ\tan 15^\circ}{\tan 55^\circ}.$$

For readability, denote $\tan 5^\circ$ by t. Then, using the difference of angles formula for tangent and simplifying, we get

$$\tan\theta = \tan 15^\circ \cdot \frac{\tan(30^\circ - 5^\circ)}{\tan(60^\circ - 5^\circ)} = \tan 15^\circ \cdot \frac{1-\sqrt{3}t}{\sqrt{3}+t} \cdot \frac{1+\sqrt{3}t}{\sqrt{3}-t} = \tan 15^\circ \cdot t \cdot \frac{1-3t^2}{3t-t^3}.$$

By the triple angle formula for tangent, $\frac{1-3t^2}{3t-t^3} = \frac{1}{\tan(3\cdot 5^\circ)}$, so it follows from the above calculation that $\tan\theta = t = \tan 5^\circ$. But $\theta \in [0, 180^\circ]$, so it follows that $\theta = 5^\circ$, as claimed.