

# THE OLYMPIAD CORNER

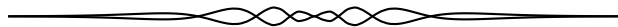
No. 328

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*Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2016** ; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

*La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.*



**OC206.** Deux cercles  $K_1$  et  $K_2$ , de rayons différents, intersectent aux points  $A$  et  $B$ . Soient  $C$  et  $D$  deux points sur  $K_1$  et  $K_2$  respectivement, tels que  $A$  est le milieu du segment  $CD$ . Le prolongement de  $DB$  rencontre  $K_1$  à un second point  $E$  et le prolongement de  $CB$  rencontre  $K_2$  à un second point  $F$ . Soient  $l_1$  et  $l_2$  les bissectrices perpendiculaires de  $CD$  et  $EF$  respectivement.

1. Démontrer que  $l_1$  et  $l_2$  ont un point commun unique, dénoté  $P$ .
2. Démontrer que les longueurs  $CA$ ,  $AP$  et  $PE$  sont les longueurs d'un triangle rectangle.

**OC207.** Déterminer toutes les fonctions injectives  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  telles que

$$|f(x) - f(y)| \leq |x - y|$$

pour tout  $x, y \in \mathbb{Z}$ .

**OC208.** Déterminer toutes les valeurs  $x$  non entières telles que

$$x + \frac{13}{x} = [x] + \frac{13}{[x]}$$

où  $[x]$  dénote le plus grand entier  $n$  plus petit ou égal à  $x$ .

**OC209.** La séquence  $a_1, a_2, \dots, a_n$  consiste des nombres  $1, 2, \dots, n$  dans un certain ordre. Pour quels entiers positifs  $n$  est-il possible que les  $n + 1$  nombres  $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n$  ont tous des restes différents lorsqu'ils sont divisés par  $n + 1$ ?

**OC210.** Déterminer tous les entiers positifs  $a$  tels que pour tout entier  $n \geq 5$ , on a  $2^n - n^2 \mid a^n - n^a$ .

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**OC206.** Two circles  $K_1$  and  $K_2$  of different radii intersect at two points  $A$  and  $B$ . Let  $C$  and  $D$  be two points on  $K_1$  and  $K_2$ , respectively, such that  $A$  is the midpoint of the segment  $CD$ . The extension of  $DB$  meets  $K_1$  at another point  $E$ , the extension of  $CB$  meets  $K_2$  at another point  $F$ . Let  $l_1$  and  $l_2$  be the perpendicular bisectors of  $CD$  and  $EF$ , respectively.

1. Show that  $l_1$  and  $l_2$  have a unique common point (denoted by  $P$ ).
2. Prove that the lengths of  $CA$ ,  $AP$  and  $PE$  are the side lengths of a right triangle.

**OC207.** Find all injective functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  that satisfy:

$$|f(x) - f(y)| \leq |x - y|$$

for any  $x, y \in \mathbb{Z}$ .

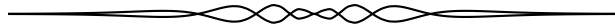
**OC208.** Find all non-integers  $x$  such that

$$x + \frac{13}{x} = [x] + \frac{13}{[x]}$$

where  $[x]$  means the greatest integer  $n$  less than or equal to  $x$ .

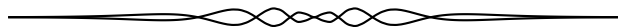
**OC209.** The sequence  $a_1, a_2, \dots, a_n$  consists of the numbers  $1, 2, \dots, n$  in some order. For which positive integers  $n$  is it possible that the  $n + 1$  numbers  $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n$  all have different remainders when divided by  $n + 1$ ?

**OC210.** Find all positive integers  $a$  such that for any positive integer  $n \geq 5$  we have  $2^n - n^2 \mid a^n - n^a$ .



## OLYMPIAD SOLUTIONS

*Les énoncés des problèmes dans cette section initialement apparaissent dans 2013: 39(10), p. 440-441.*



**OC146.** Let  $ABC$  be an isosceles triangle with  $AC = BC$ . Take points  $D$  on side  $AC$  and  $E$  on side  $BC$ . Let  $F$  be the intersection of bisectors of angles  $DEB$  and  $ADE$ . If  $F$  lies on side  $AB$ , prove that  $F$  is the midpoint of  $AB$ .

*Originally from Moldova Junior Balkan Team Selection Test Day 2, Problem 3.*

*We received seven correct submissions. We present the solution by Adnan Ali.*

Let  $d(X, PQ)$  denote the perpendicular distance of point  $X$  from line  $PQ$ . Then we observe that since  $F$  lies on the angle bisector of  $\angle ADE$ ,

$$d(F, DE) = d(F, DA) = d(F, AC).$$

And since  $F$  lies on the angle bisector of  $\angle DEB$ , we also have

$$d(F, DE) = d(F, EB) = d(F, BC).$$

This shows that  $F$  is equidistant from  $AC$  and  $BC$  and so it must lie on the angle bisector of  $\angle ACB$ . Therefore  $FC$  is the bisector of the angle  $ACB$ . As  $AC = BC$ , it follows that  $FC$  is also the midline of the triangle, and hence  $F$  is the midpoint of  $AB$ .

**OC147.** Suppose  $a_1$  is a natural number and  $\{a_n\}_n$ , is defined by the rule:

$$a_{n+1} = a_n + 2d(n),$$

where  $d(n)$  denotes the number of different divisors of  $n$  (including 1 and  $n$ ). Does there exist an  $a_1$  such that two consecutive members of the sequence are squares of natural numbers?

*Originally from the Bulgaria Mathematical Olympiad Day 1, Problem 2.*

*We present the solution by Oliver Geupel. There were no other submitted solutions.*

We show that the answer is No.

Suppose by contradiction that, for some properly chosen initial value  $a_1 \geq 1$ , the members  $a_n$  and  $a_{n+1}$  are perfect squares. Let  $a_n = m^2$ .

Since  $a_{n+1} - a_n = 2d(n)$  and  $(m+1)^2 - m^2 = 2m+1$  is odd, we cannot have  $a_{n+1} = (m+1)^2$ .

Therefore,  $a_{n+1} \geq (m+2)^2$ , and hence

$$4m+4 = (m+2)^2 - m^2 \leq a_{n+1} - a_n = 2d(n) \leq 4\sqrt{n}.$$

This implies

$$(m+1)^2 \leq n.$$

Since,  $a_1 \geq 1$ ,  $d(1) = 1$ , and  $d(k) \geq 2$  for  $k \geq 2$ , by induction we have  $a_n \geq 4n - 5$ .

Therefore

$$m^2 = a_n \geq 4n - 5 \geq 4(m+1)^2 - 5,$$

and hence  $3m^3 + 8m \leq 1$ , which is a contradiction. This shows that our assumption is wrong, and therefore there is no value of  $a_1$  for which two consecutive terms of the sequence are perfect squares.

**OC148.** Complex numbers  $x_i, y_i$  satisfy  $|x_i| = |y_i| = 1$  for all  $1 \leq i \leq n$ . Let  $x = \frac{1}{n} \sum_{i=1}^n x_i$ ,  $y = \frac{1}{n} \sum_{i=1}^n y_i$  and  $z_i = xy_i + yx_i - x_i y_i$ . Prove that

$$\sum_{i=1}^n |z_i| \leq n.$$

*Originally from the China Team Selection Test 2012, Problem 1.*

*We received two correct submissions. We present the solution by Michel Bataille.*

For each  $i$  such that  $1 \leq i \leq n$ , we have  $z_i = xy - (x_i - x)(y_i - y)$ , hence

$$|z_i| \leq |x||y| + |x_i - x||y_i - y| \leq \frac{|x|^2 + |y|^2}{2} + \frac{|x_i - x|^2 + |y_i - y|^2}{2}.$$

It follows that

$$\sum_{i=1}^n |z_i| \leq \frac{1}{2} \left( n|x|^2 + \sum_{i=1}^n |x_i - x|^2 + n|y|^2 + \sum_{i=1}^n |y_i - y|^2 \right). \quad (1)$$

Now, from

$$\begin{aligned} |x_i|^2 &= |(x_i - x) + x|^2 = ((x_i - x) + x) \cdot \overline{((x_i - x) + x)} \\ &= |x_i - x|^2 + (x_i - x)\bar{x} + \overline{(x_i - x)}x + |x|^2, \end{aligned}$$

we deduce

$$\sum_{i=1}^n |x_i|^2 = \sum_{i=1}^n |x_i - x|^2 + \bar{x} \sum_{i=1}^n (x_i - x) + x \sum_{i=1}^n \overline{(x_i - x)} + n|x|^2 = n|x|^2 + \sum_{i=1}^n |x_i - x|^2,$$

with the last equality following from

$$\sum_{i=1}^n (x_i - x) = \left( \sum_{i=1}^n x_i \right) - nx = 0.$$

A similar result holds for  $\sum_{i=1}^n |y_i|^2$  and returning to (1), we obtain

$$\sum_{i=1}^n |z_i| \leq \frac{1}{2} \left( \sum_{i=1}^n |x_i|^2 + \sum_{i=1}^n |y_i|^2 \right).$$

The required result follows now as  $|x_i| = |y_i| = 1$ .

**OC149.** Find all functions  $f: \mathbb{Z} \rightarrow \mathbb{Z}$ , such that for all integers  $x, y$  we have

$$f(x + y + f(y)) = f(x) + ny,$$

where  $n$  is a fixed integer.

*Originally from China TST 2012 Day 1, Problem 3.*

*We received two correct submissions. We present the solution by Oliver Geupel.*

It is straightforward to verify that the two functions

$$f(x) = mx \quad \text{and} \quad f(x) = -(m+1)x$$

are solutions of the given functional equation if the number  $n$  can be written in the form

$$n = m(m+1)$$

with an integer  $m$ . We prove that there are no further solutions.

We show that if  $f$  is any solution of the given functional equation, then  $f(x) = lx$  for some integer  $l$  so that  $l(l+1) = n$ . Let  $f$  be a solution to the equation.

For  $k \in \mathbb{Z}$ , let  $P(k)$  denote the assertion that the identity

$$f(k(x + f(x))) = f(0) + knx$$

holds for every integer  $x$ . We prove that  $P(k)$  holds for every  $k \in \mathbb{Z}$ . First, we show  $P(k)$  for  $k \geq 0$  by induction on  $k$ .

The base case  $k = 0$  is obvious.

Assuming  $P(k)$ , we obtain

$$\begin{aligned} f((k+1)(x + f(x))) &= f(k(x + f(x)) + x + f(x)) = f(k(x + f(x))) + nx \\ &= f(0) + knx + nx = f(0) + (k+1)nx, \end{aligned}$$

that is,  $P(k+1)$ , which completes the induction.

It remains to show  $P(-k)$  for  $k \geq 0$  by induction on  $k$ . Assume  $P(-k)$ . Then,

$$\begin{aligned} f(0) - knx &= f(-k(x + f(x))) = f(-(k+1)(x + f(x)) + x + f(x)) \\ &= f(-(k+1)(x + f(x))) + nx; \end{aligned}$$

whence  $f(-(k+1)(x+f(x))) = f(0) - (k+1)nx$ , which is  $P(-(k+1))$ , thus completing the induction. Thus  $P(k)$  holds for every integer  $k$ .

As a consequence, we successively obtain

$$f(0) + (x+f(x))ny = f((x+f(x))(y+f(y))) = f(0) + (y+f(y))nx$$

and

$$xf(y) = yf(x)$$

for all integers  $x$  and  $y$ . Setting  $y = 1$  gives

$$f(x) = xf(1) = lx.$$

where  $l = f(1)$ . Plugging this into the original functional equation, we obtain

$$f(1)(x+y+yf(1)) = f(x+y+f(y)) = f(x) + ny = xf(1) + ny.$$

Therefore

$$n = f(1)(f(1) + 1) = l(l + 1).$$

This completes the proof that there are no further solutions.

**OC150.** Let  $ABC$  be an isosceles triangle with  $AB = AC$  and let  $D$  be the leg of perpendicular from  $A$ .  $P$  is an interior point of triangle  $ADC$  such that  $\angle APB > 90^\circ$  and  $\angle PBD + \angle PAD = \angle PCB$ . Let  $Q$  be the intersection of  $CP$  and  $AD$ , and let  $R$  be the intersection of  $BP$  and  $AD$ . Let  $T$  be a point on  $[AB]$  and let  $S$  be a point on  $[AP]$  not belonging to  $[AP]$  such that  $\angle TRB = \angle DQC$  and  $\angle PSR = 2\angle PAR$ . Prove that  $RS = RT$ .

*Originally from Turkey National Olympiad Second Round 2012, Day 1, Problem 2.*

*We received two correct submissions. We present the solution by Oliver Geupel.*

Fixing the ambiguity in the problem statement, suppose that  $S$  is a point on the half-line from point  $A$  through point  $P$ , not belonging to the segment  $[AP]$ .

Let

$$\alpha = \angle QBA, \beta = \angle RBQ \text{ and } \gamma = \angle DBR.$$

The following relations are found by routine angle chasing:

$$\begin{aligned} \angle PAC &= 90^\circ - \alpha - 2\beta - \gamma, & \angle PBA &= \alpha + \gamma, \\ \angle BAP &= 90^\circ - \alpha - \gamma, & \angle CBP &= \gamma, \\ \angle PCB &= \beta + \gamma, & \angle ACP &= \alpha \end{aligned}$$

By the trigonometric version of Ceva's theorem, for the triangle  $ABC$  and the point  $P$  it holds

$$\begin{aligned} \frac{\cos(\alpha + 2\beta + \gamma)}{\cos(\alpha + \gamma)} \cdot \frac{\sin(\alpha + \beta)}{\sin \gamma} \cdot \frac{\sin(\beta + \gamma)}{\sin \alpha} \\ = \frac{\sin \angle PAC}{\sin \angle BAP} \cdot \frac{\sin \angle PBA}{\sin \angle CBP} \cdot \frac{\sin \angle PCB}{\sin \angle ACP} = 1. \end{aligned} \quad (1)$$

and for the triangle  $ATS$  and the point  $R$  we have

$$\frac{\sin \beta}{\cos(\alpha + \beta + \gamma)} \cdot \frac{\cos(\gamma - \alpha)}{\sin \angle STR} \cdot \frac{\sin \angle RST}{\sin 2\beta} = 1. \quad (2)$$

From (1) we deduce

$$\begin{aligned} 0 &= \cos(\alpha + 2\beta + \gamma) \sin(\alpha + \beta) \sin(\beta + \gamma) - \cos(\alpha + \gamma) \sin \gamma \sin \alpha \\ &= \frac{1}{4} (\cos(2\alpha + 2\beta) + \cos(2\beta + 2\gamma) + \cos(2\gamma + 2\alpha) - \cos 2\alpha - \cos 2\gamma \\ &\quad - \cos(2\alpha + 4\beta + 2\gamma)) \\ &= \frac{1}{2} (\cos(\alpha + \gamma) - \cos(\alpha + 2\beta + \gamma)) (\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma)) \\ &= \sin \beta \sin(\alpha + \beta + \gamma) (\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma)), \end{aligned}$$

whence

$$\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma) = 0.$$

Therefore,

$$\begin{aligned} \sin \beta \cos(\gamma - \alpha) - \cos(\alpha + \beta + \gamma) \sin 2\beta \\ = -\sin \beta (2 \cos(\alpha + \beta + \gamma) \cos \beta - \cos(\alpha - \gamma)) \\ = -\sin \beta (\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma)) \\ = 0. \end{aligned}$$

By (2), we conclude that  $\angle RST = \angle STR$ , that is, the triangle  $RST$  is isosceles with  $RS = RT$ . The proof is complete.

