OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section initialement apparaissent dans 2013: 39(10), p. 440–441.

OC146. Let ABC be an isosceles triangle with $AC = BC$. Take points D on side AC and E on side BC. Let F be the intersection of bisectors of angles DEB and ADE. If F lies on side AB, prove that F is the midpoint of AB.

Originally from Moldova Junior Balkan Team Selection Test Day 2, Problem 3.

We received seven correct submissions. We present the solution by Adnan Ali.

Let $d(X, PQ)$ denote the perpendicular distance of point X from line PQ. Then we observe that since F lies on the angle bisector of $\angle ADE$,

$$d(F, DE) = d(F, DA) = d(F, AC).$$

And since F lies on the angle bisector of $\angle DEB$, we also have

$$d(F, DE) = d(F, EB) = d(F, BC).$$

This shows that F is equidistant from AC and BC and so it must lie on the angle bisector of $\angle ACB$. Therefore FC is the bisector of the angle ACB. As $AC = BC$, it follows that FC is also the midline of the triangle, and hence F is the midpoint of AB.

OC147. Suppose a_1 is a natural number and $\{a_n\}_n$, is defined by the rule:

$$a_{n+1} = a_n + 2d(n)\,,$$

where $d(n)$ denotes the number of different divisors of n (including 1 and n). Does there exist an a_1 such that two consecutive members of the sequence are squares of natural numbers?

Originally from the Bulgaria Mathematical Olympiad Day 1, Problem 2.

We present the solution by Oliver Geupel. There were no other submitted solutions.

We show that the answer is No.

Suppose by contradiction that, for some properly chosen initial value $a_1 \geq 1$, the members a_n and a_{n+1} are perfect squares. Let $a_n = m^2$.

Since $a_{n+1} - a_n = 2d(n)$ and $(m+1)^2 - m^2 = 2m + 1$ is odd, we cannot have $a_{n+1} = (m+1)^2$.

Therefore, $a_{n+1} \geq (m+2)^2$, and hence

$$4m + 4 = (m+2)^2 - m^2 \leq a_{n+1} - a_n = 2d(n) \leq 4\sqrt{n}.$$

This implies
$$(m+1)^2 \le n\,.$$

Since, $a_1 \ge 1$, $d(1)=1$, and $d(k) \ge 2$ for $k \ge 2$, by induction we have $a_n \ge 4n-5$. Therefore
$$m^2 = a_n \ge 4n-5 \ge 4(m+1)^2 - 5,$$
and hence $3m^3+8m \le 1$, which is a contradiction. This shows that our assumption is wrong, and therefore there is no value of a_1 for which two consecutive terms of the sequence are perfect squares.

OC148. Complex numbers x_i, y_i satisfy $|x_i| = |y_i| = 1$ for all $1 \le i \le n$. Let $x = \frac{1}{n}\sum_{i=1}^{n} x_i$, $y = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $z_i = xy_i + yx_i - x_iy_i$. Prove that
$$\sum_{i=1}^{n} |z_i| \le n\,.$$

Originally from the China Team Selection Test 2012, Problem 1.

We received two correct submissions. We present the solution by Michel Bataille.

For each i such that $1 \le i \le n$, we have $z_i = xy - (x_i - x)(y_i - y)$, hence
$$|z_i| \le |x||y| + |x_i - x||y_i - y| \le \frac{|x|^2+|y|^2}{2} + \frac{|x_i-x|^2+|y_i-y|^2}{2}\,.$$

It follows that
$$\sum_{i=1}^{n} |z_i| \le \frac{1}{2}\left(n|x|^2 + \sum_{i=1}^{n}|x_i-x|^2 + n|y|^2 + \sum_{i=1}^{n}|y_i-y|^2\right)\,. \qquad (1)$$

Now, from
$$\begin{aligned}|x_i|^2 = |(x_i-x)+x|^2 &= ((x_i-x)+x)\cdot\left(\overline{(x_i-x)}+\overline{x}\right)\\ &= |x_i-x|^2 + (x_i-x)\overline{x} + \overline{(x_i-x)}x + |x|^2,\end{aligned}$$
we deduce
$$\sum_{i=1}^{n}|x_i|^2 = \sum_{i=1}^{n}|x_i-x|^2 + \overline{x}\sum_{i=1}^{n}(x_i-x) + x\overline{\sum_{i=1}^{n}(x_i-x)} + n|x|^2 = n|x|^2 + \sum_{i=1}^{n}|x_i-x|^2\,,$$
with the last equality following from
$$\sum_{i=1}^{n}(x_i-x) = \Big(\sum_{i=1}^{n}x_i\Big) - nx = 0\,.$$

A similar result holds for $\sum_{i=1}^{n} |y_i|^2$ and returning to (1), we obtain

$$\sum_{i=1}^{n} |z_i| \le \frac{1}{2} \left(\sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2 \right) .$$

The required result follows now as $|x_i| = |y_i| = 1$.

OC149. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have

$$f(x + y + f(y)) = f(x) + ny ,$$

where n is a fixed integer.

Originally from China TST 2012 Day 1, Problem 3.

We received two correct submissions. We present the solution by Oliver Geupel.

It is straightforward to verify that the two functions

$$f(x) = mx \qquad \text{and} \qquad f(x) = -(m+1)x$$

are solutions of the given functional equation if the number n can be written in the form

$$n = m(m+1)$$

with an integer m. We prove that there are no further solutions.

We show that if f is any solution of the given functional equation, then $f(x) = lx$ for some integer l so that $l(l+1) = n$. Let f be a solution to the equation.

For $k \in \mathbb{Z}$, let $P(k)$ denote the assertion that the identity

$$f(k(x + f(x))) = f(0) + knx$$

holds for every integer x. We prove that $P(k)$ holds for every $k \in \mathbb{Z}$. First, we show $P(k)$ for $k \ge 0$ by induction on k.

The base case $k = 0$ is obvious.

Assuming $P(k)$, we obtain

$$\begin{aligned} f((k+1)(x + f(x))) &= f(k(x + f(x)) + x + f(x)) = f(k(x + f(x))) + nx \\ &= f(0) + knx + nx = f(0) + (k+1)nx , \end{aligned}$$

that is, $P(k+1)$, which completes the induction.

It remains to show $P(-k)$ for $k \ge 0$ by induction on k. Assume $P(-k)$. Then,

$$\begin{aligned} f(0) - knx &= f(-k(x + f(x))) = f(-(k+1)(x + f(x)) + x + f(x)) \\ &= f(-(k+1)(x + f(x))) + nx ; \end{aligned}$$

whence $f(-(k+1)(x+f(x))) = f(0) - (k+1)nx$, which is $P(-(k+1))$, thus completing the induction. Thus $P(k)$ holds for every integer k.

As a consequence, we successively obtain

$$f(0) + (x + f(x))ny = f((x + f(x))(y + f(y))) = f(0) + (y + f(y))nx$$

and

$$xf(y) = yf(x)$$

for all integers x and y. Setting $y = 1$ gives

$$f(x) = xf(1) = lx\,.$$

where $l = f(1)$. Plugging this into the original functional equation, we obtain

$$f(1)(x + y + yf(1)) = f(x + y + f(y)) = f(x) + ny = xf(1) + ny.$$

Therefore

$$n = f(1)(f(1) + 1) = l(l + 1).$$

This completes the proof that there are no further solutions.

OC150. Let ABC be an isosceles triangle with $AB = AC$ and let D be the leg of perpendicular from A. P is an interior point of triangle ADC such that $\angle APB > 90^\circ$ and $\angle PBD + \angle PAD = \angle PCB$. Let Q be the intersection of CP and AD, and let R be the intersection of BP and AD. Let T be a point on $[AB]$ and let S be a point on $[AP]$ not belonging to $[AP]$ such that $\angle TRB = \angle DQC$ and $\angle PSR = 2\angle PAR$. Prove that $RS = RT$.

Originally from Turkey National Olympiad Second Round 2012, Day 1, Problem 2.

We received two correct submissions. We present the solution by Oliver Geupel.

Fixing the ambiguity in the problem statement, suppose that S is a point on *the half-line from point A through point P*, not belonging to the segment $[AP]$.

Let

$$\alpha = \angle QBA\,,\ \beta = \angle RBQ \ \text{ and } \ \gamma = \angle DBR\,.$$

The following relations are found by routine angle chasing:

$$\begin{aligned} \angle PAC &= 90^\circ - \alpha - 2\beta - \gamma\,, & \angle PBA &= \alpha + \gamma\,, \\ \angle BAP &= 90^\circ - \alpha - \gamma\,, & \angle CBP &= \gamma\,, \\ \angle PCB &= \beta + \gamma\,, & \angle ACP &= \alpha \end{aligned}$$

By the trigonometric version of Ceva's theorem, for the triangle ABC and the point P it holds

$$\begin{aligned}&\frac{\cos(\alpha+2\beta+\gamma)}{\cos(\alpha+\gamma)}\cdot\frac{\sin(\alpha+\beta)}{\sin\gamma}\cdot\frac{\sin(\beta+\gamma)}{\sin\alpha}\\&\qquad=\frac{\sin\angle PAC}{\sin\angle BAP}\cdot\frac{\sin\angle PBA}{\sin\angle CBP}\cdot\frac{\sin\angle PCB}{\sin\angle ACP}=1\,.\end{aligned}\tag{1}$$

and for the triangle ATS and the point R we have

$$\frac{\sin\beta}{\cos(\alpha+\beta+\gamma)}\cdot\frac{\cos(\gamma-\alpha)}{\sin\angle STR}\cdot\frac{\sin\angle RST}{\sin 2\beta}=1\,.\tag{2}$$

From (1) we deduce

$$\begin{aligned}0&=\cos(\alpha+2\beta+\gamma)\sin(\alpha+\beta)\sin(\beta+\gamma)-\cos(\alpha+\gamma)\sin\gamma\sin\alpha\\&=\frac{1}{4}\,(\cos(2\alpha+2\beta)+\cos(2\beta+2\gamma)+\cos(2\gamma+2\alpha)-\cos 2\alpha-\cos 2\gamma\\&\qquad\qquad-\cos(2\alpha+4\beta+2\gamma))\\&=\frac{1}{2}\big(\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\big)\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big)\\&=\sin\beta\sin(\alpha+\beta+\gamma)\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big),\end{aligned}$$

whence

$$\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)=0.$$

Therefore,

$$\begin{aligned}&\sin\beta\cos(\gamma-\alpha)-\cos(\alpha+\beta+\gamma)\sin 2\beta\\&=-\sin\beta\big(2\cos(\alpha+\beta+\gamma)\cos\beta-\cos(\alpha-\gamma)\big)\\&=-\sin\beta\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big)\\&=0\,.\end{aligned}$$

By (2), we conclude that $\angle RST=\angle STR$, that is, the triangle RST is isosceles with $RS=RT$. The proof is complete.