OLYMPIAD SOLUTIONS


**OC146.** Let $ABC$ be an isosceles triangle with $AC = BC$. Take points $D$ on side $AC$ and $E$ on side $BC$. Let $F$ be the intersection of bisectors of angles $DEB$ and $ADE$. If $F$ lies on side $AB$, prove that $F$ is the midpoint of $AB$.

*Originally from Moldova Junior Balkan Team Selection Test Day 2, Problem 3.*

We received seven correct submissions. We present the solution by Adnan Ali.

Let $d(X,PQ)$ denote the perpendicular distance of point $X$ from line $PQ$. Then we observe that since $F$ lies on the angle bisector of $∠ADE$,

$$d(F,DE) = d(F,DA) = d(F,AC).$$

And since $F$ lies on the angle bisector of $∠DEB$, we also have

$$d(F,DE) = d(F,EB) = d(F,BC).$$

This shows that $F$ is equidistant from $AC$ and $BC$ and so it must lie on the angle bisector of $∠ACB$. Therefore $FC$ is the bisector of the angle $ACB$. As $AC = BC$, it follows that $FC$ is also the midline of the triangle, and hence $F$ is the midpoint of $AB$.

**OC147.** Suppose $a_1$ is a natural number and $(a_n)_n$, is defined by the rule:

$$a_{n+1} = a_n + 2d(n),$$

where $d(n)$ denotes the number of different divisors of $n$ (including 1 and $n$). Does there exist an $a_1$ such that two consecutive members of the sequence are squares of natural numbers?

*Originally from the Bulgaria Mathematical Olympiad Day 1, Problem 2.*

*We present the solution by Oliver Geupel. There were no other submitted solutions.*

We show that the answer is No.

Suppose by contradiction that, for some properly chosen initial value $a_1 \geq 1$, the members $a_n$ and $a_{n+1}$ are perfect squares. Let $a_n = m^2$.

Since $a_{n+1} - a_n = 2d(n)$ and $(m + 1)^2 - m^2 = 2m + 1$ is odd, we cannot have $a_{n+1} = (m + 1)^2$.

Therefore, $a_{n+1} \geq (m + 2)^2$, and hence

$$4m + 4 = (m + 2)^2 - m^2 \leq a_{n+1} - a_n = 2d(n) \leq 4\sqrt{n}.$$
This implies

\[(m + 1)^2 \leq n.\]

Since, \(a_1 \geq 1, d(1) = 1,\) and \(d(k) \geq 2\) for \(k \geq 2,\) by induction we have \(a_n \geq 4n - 5.\)

Therefore

\[m^2 = a_n \geq 4n - 5 \geq 4(m + 1)^2 - 5,
\]

and hence \(3m^2 + 8m \leq 1,\) which is a contradiction. This shows that our assumption is wrong, and therefore there is no value of \(a_1\) for which two consecutive terms of the sequence are perfect squares.

**OC148.** Complex numbers \(x_i, y_i\) satisfy \(|x_i| = |y_i| = 1\) for all \(1 \leq i \leq n.\) Let

\[x = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad y = \frac{1}{n} \sum_{i=1}^{n} y_i\]

and \(z_i = xy_i + yx_i - x_iy_i.\) Prove that

\[\sum_{i=1}^{n} |z_i| \leq n.
\]

*Originally from the China Team Selection Test 2012, Problem 1.*

We received two correct submissions. We present the solution by Michel Bataille.

For each \(i\) such that \(1 \leq i \leq n,\) we have \(z_i = xy - (x_i - x)(y_i - y),\) hence

\[|z_i| \leq |x||y| + |x_i - x||y_i - y| \leq \frac{|x|^2 + |y|^2}{2} + \frac{|x - x_i|^2 + |y - y_i|^2}{2}.
\]

It follows that

\[
\sum_{i=1}^{n} |z_i| \leq \frac{1}{2} \left(n|x|^2 + \sum_{i=1}^{n} |x_i - x|^2 + n|y|^2 + \sum_{i=1}^{n} |y_i - y|^2\right). \tag{1}
\]

Now, from

\[|x_i|^2 = |(x_i - x) + x|^2 = ((x_i - x) + x) \cdot ((x_i - x) + x) = |x_i - x|^2 + (x_i - x)x + (x_i - x)x + |x|^2,
\]

we deduce

\[
\sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} |x_i - x|^2 + \sum_{i=1}^{n} (x_i - x) + x \sum_{i=1}^{n} (x_i - x) + n|x|^2 = n|x|^2 + \sum_{i=1}^{n} |x_i - x|^2,
\]

with the last equality following from

\[
\sum_{i=1}^{n} (x_i - x) = (\sum_{i=1}^{n} x_i) - nx = 0.
\]

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A similar result holds for \( \sum_{i=1}^{n} |y_i|^2 \) and returning to (1), we obtain
\[
\sum_{i=1}^{n} |z_i| \leq \frac{1}{2} \left( \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2 \right).
\]
The required result follows now as \( |x_i| = |y_i| = 1 \).

**OC149.** Find all functions \( f: \mathbb{Z} \to \mathbb{Z} \), such that for all integers \( x, y \) we have
\[
f(x + y + f(y)) = f(x) + ny ,
\]
where \( n \) is a fixed integer.

*Originally from China TST 2012 Day 1, Problem 3.*

We received two correct submissions. We present the solution by Oliver Geupel.

It is straightforward to verify that the two functions
\[
f(x) = mx \quad \text{and} \quad f(x) = -(m+1)x
\]
are solutions of the given functional equation if the number \( n \) can be written in the form
\[
n = m(m+1)
\]
with an integer \( m \). We prove that there are no further solutions.

We show that if \( f \) is any solution of the given functional equation, then \( f(x) = lx \) for some integer \( l \) so that \( l(l+1) = n \). Let \( f \) be a solution to the equation.

For \( k \in \mathbb{Z} \), let \( P(k) \) denote the assertion that the identity
\[
f(k(x + f(x))) = f(0) + knx
\]
holds for every integer \( x \). We prove that \( P(k) \) holds for every \( k \in \mathbb{Z} \). First, we show \( P(k) \) for \( k \geq 0 \) by induction on \( k \).

The base case \( k = 0 \) is obvious.

Assuming \( P(k) \), we obtain
\[
f((k+1)(x + f(x))) = f(kx + f(x)) + x + f(x) = f(k(x + f(x))) + nx
\]
\[
= f(0) + knx + nx = f(0) + (k+1)nx ,
\]
that is, \( P(k+1) \), which completes the induction.

It remains to show \( P(-k) \) for \( k \geq 0 \) by induction on \( k \). Assume \( P(-k) \). Then,
\[
f(0) - knx = f(-(k+1)(x + f(x))) = f(-(k+1)(x + f(x)) + x + f(x))
\]
\[
= f(-(k+1)(x + f(x))) + nx ;
\]

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whence \( f(-(k+1)(x+f(x))) = f(0) - (k+1)nx \), which is \( P(-(k+1)) \), thus completing the induction. Thus \( P(k) \) holds for every integer \( k \).

As a consequence, we successively obtain

\[
 f(0) + (x + f(x))ny = f((x + f(x))(y + f(y))) = f(0) + (y + f(y))nx
\]

and

\[
xf(y) = yf(x)
\]

for all integers \( x \) and \( y \). Setting \( y = 1 \) gives

\[
f(x) = xf(1) = lx.
\]

where \( l = f(1) \). Plugging this into the original functional equation, we obtain

\[
f(1)(x + y + yf(1)) = f(x + y + f(y)) = f(x) + ny = xf(1) + ny.
\]

Therefore

\[
n = f(1)(f(1) + 1) = l(l + 1).
\]

This completes the proof that there are no further solutions.

**OC150.** Let \( ABC \) be an isosceles triangle with \( AB = AC \) and let \( D \) be the leg of perpendicular from \( A \). \( P \) is an interior point of triangle \( ADC \) such that \( \angle APB > 90^\circ \) and \( \angle PBD + \angle PAD = \angle PCB \). Let \( Q \) be the intersection of \( CP \) and \( AD \), and let \( R \) be the intersection of \( BP \) and \( AD \). Let \( T \) be a point on \([AB]\) and let \( S \) be a point on \([AP]\) not belonging to \([AP]\) such that \( \angle TRB = \angle DQC \) and \( \angle PSR = 2\angle PAR \). Prove that \( RS = RT \).

*Originally from Turkey National Olympiad Second Round 2012, Day 1, Problem 2.*

We received two correct submissions. We present the solution by Oliver Geupel.

Fixing the ambiguity in the problem statement, suppose that \( S \) is a point on the half-line from point \( A \) through point \( P \), not belonging to the segment \([AP]\).

Let

\[
\alpha = \angle QBA, \beta = \angle RBQ \text{ and } \gamma = \angle DBR.
\]

The following relations are found by routine angle chasing:

\[
\angle PAC = 90^\circ - \alpha - 2\beta - \gamma, \quad \angle PBA = \alpha + \gamma,
\]

\[
\angle BAP = 90^\circ - \alpha - \gamma, \quad \angle CBP = \gamma,
\]

\[
\angle PCB = \beta + \gamma, \quad \angle ACP = \alpha
\]

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By the trigonometric version of Ceva’s theorem, for the triangle $ABC$ and the point $P$ it holds

\[
\frac{\cos(\alpha + 2\beta + \gamma)}{\cos(\alpha + \gamma)} \cdot \frac{\sin(\alpha + \beta)}{\sin \gamma} \cdot \frac{\sin(\beta + \gamma)}{\sin \alpha} = \frac{\sin \angle PAC}{\sin \angle BAP} \cdot \frac{\sin \angle PBA}{\sin \angle CBA} \cdot \frac{\sin \angle PCB}{\sin \angle ACP} = 1.
\] (1)

and for the triangle $ATS$ and the point $R$ we have

\[
\frac{\sin \beta}{\cos(\alpha + \beta + \gamma)} \cdot \frac{\cos(\gamma - \alpha)}{\sin \angle STR} \cdot \frac{\sin \angle RST}{\sin 2\beta} = 1.
\] (2)

From (1) we deduce

\[
0 = \cos(\alpha + 2\beta + \gamma) \sin(\alpha + \beta) \sin(\beta + \gamma) - \cos(\alpha + \gamma) \sin \gamma \sin \alpha
= \frac{1}{4} (\cos(2\alpha + 2\beta) + \cos(2\beta + 2\gamma) + \cos(2\gamma + 2\alpha) - \cos 2\alpha - \cos 2\gamma
- \cos(2\alpha + 4\beta + 2\gamma))
= \frac{1}{2} (\cos(\alpha + \gamma) - \cos(\alpha + 2\beta + \gamma)) (\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma))
= \sin \beta \sin(\alpha + \beta + \gamma)(\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma)),
\]
whence

\[
\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma) = 0.
\]

Therefore,

\[
\sin \beta \cos(\gamma - \alpha) - \cos(\alpha + \beta + \gamma) \sin 2\beta
= -\sin \beta (2 \cos(\alpha + \beta + \gamma) \cos \beta - \cos(\alpha - \gamma))
= -\sin \beta (\cos(\alpha + 2\beta + \gamma) - \cos(\alpha - \gamma) + \cos(\alpha + \gamma))
= 0.
\]

By (2), we conclude that $\angle RST = \angle STR$, that is, the triangle $RST$ is isosceles with $RS = RT$. The proof is complete.