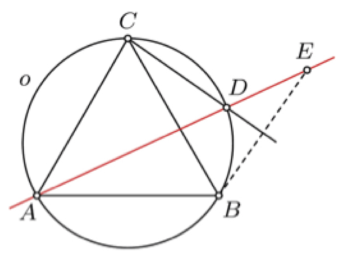
CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent dans 2013 : 39(10), p. 435–436.

CC96. An equilateral triangle ABC is inscribed in a circle o. Point D is on arc BC of o. Point E is the symmetric of B with respect to line CD. Prove that A, D and E are collinear.



Originally 10th secondary mathematics olympiad (Poland), First level, question 2.

We received two correct solutions, and one incorrect solution. We present the solution of John Heuver, slightly modified by the editor.

Quadrilateral $ABDC$ is cyclic, which implies that $\measuredangle BDC = 180 - \measuredangle CAB = 120°$. By symmetry, $\measuredangle BDC = \measuredangle EDC = 120°$, and hence $\measuredangle BDE = 120°$. Further, $\measuredangle ACB = \measuredangle ADB = 60°$, since the two angles are subtended by the same arc. Thus $\measuredangle ADB + \measuredangle BDE = 60° + 120° = 180°$; that is, A, D and E are collinear.

CC97. Find the smallest value of the expression $a+b^3$ where a and b are positive numbers whose product is 1.

Originally 10th secondary mathematics olympiad (Poland), First level, question 3.

We received nine correct submissions and one incorrect submission. We present the solution by Henry Ricardo.

Using the fact that $ab = 1$, the Arithmetic-Geometric Mean inequality yields

$$a + b^3 \; = \; a + \frac{1}{a^3} \; = \; \frac{a}{3} + \frac{a}{3} + \frac{a}{3} + \frac{1}{a^3} \; \geq \; 4\sqrt[4]{\frac{a^3}{27a^3}} \; = \; \frac{4}{\sqrt[4]{27}} \; \approx \; 1.754765351.$$

This minimum value is attained when $\dfrac{a}{3} = \dfrac{1}{a^3}$: that is, when $a = \sqrt[4]{3}$.

CC98. Are there real numbers x and y such that $\sqrt{x^2+1} + \sqrt{y^2+1} = x + y$?

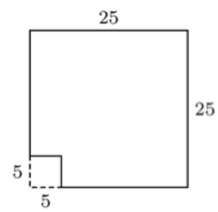
Originally 7th secondary mathematics olympiad (Poland), First level, question 1.

We received seven correct submissions. We present the solution by Henry Ricardo.

The answer is no.

We have $\sqrt{x^2+1}+\sqrt{y^2+1} > \sqrt{x^2}+\sqrt{y^2} = |x|+|y| \geq x+y$.

CC99. We cut off a square of side 5 from the corner of a square of side 25. Can we cut the remaining part into 100 rectangles of dimension either 1×6 or 2×3 ?



Originally 10th secondary mathematics olympiad (Poland), First level, question 6.

We present the solution by Titu Zvonaru.

It is not possible. Consider the original square of side length 25 as a board with 625 cells of side length 1. Colour the cells with three colours red, white, and blue such that the bottom left corner is coloured red, and such that the colours red, white, and blue alternate in that order along each row (from left to right) and each column (from bottom to top). If we cut out the bottom left square of side length 5, the remaining board contains 201 red cells, 199 white cells, and 200 blue cells. As any 1×6 or 2×3 rectangle covers exactly 2 red, 2 white, and 2 blue cells, it is not possible to cut this board into 100 such rectangles.

CC100. In a 6-team tournament, each team played with each other team exactly once. A team gets 3 points for a victory, 1 point for a draw and 0 for a defeat. After the tournament, the sum of the scores by all the teams is 41. Prove that there exists a group of 4 teams where each team tied at least once.

Originally 7th secondary mathematics olympiad (Poland), Second level, question 2.

We received three correct submissions. We present the solution by Titu Zvonaru.

Let x be the number of games that end in a draw. Since 3 total points are awarded for a win and 2 total points for a tie, we obtain

$$2x+3(15-x)=41.$$

Solving yields $x=4$, so there were 4 games which ended in a draw. Amongst 3 teams a total of 3 games are played, so there must have been at least 4 teams that were involved in ties, so there is a group of 4 teams where each team tied at least once.