26th Tournament of Towns : A Square from Similar Rectangles

Olga Zaitseva

Seventy of the brightest young people from different parts of the world were brought together to spend nine days immersing themselves in current streams of mathematical research : the 26th Tournament of Towns Summer Conference took place August 2–11, 2014, in Russia. The conference was held at the Centre for the Gifted Children, 20 km away from Kaliningrad (former Königsberg).

The location of the Summer Conferences changes every year but the structure stays the same. For each conference, a group of world-renowned mathematicians and educators (authors of projects, called the Jury) creates several topics for investigation. No topic has ever been repeated. After the participants arrive to the conference, the authors of projects give orientation lectures and hand out materials on each topic. Each participant then chooses one project to work on and authors of projects become the participants' mentors.

Although the informal contacts between participants and mentors are ongoing, there is a designated date when participants need to submit their solutions. The Jury checks the progress, removes solved parts, and supplies new information for the projects. If needed, the members of the Jury deliver lectures on methods that could be used in the research. After the deadline, authors of projects analyze the results, evaluate the progress of participants and overall teams' progress and designate directions in which the work can be continued.

This year, the Jury devised seven topics for investigation – a record number. The list includes : *A Square from Similar Rectangles, Algorithms and Labyrinths, Division of Segment, Combinatorial Geometry and Graph Colourings : from Algebra to Probability, De Bruijn Sequences and Universal Cycles, On the Poncelet Theorem*, and *Point-Line Incidences*.

Let us consider one of the topics, *A Square from Similar Rectangles*, in more detail. This topic has received much interest lately due to revealed connections with physics, harmonic analysis and the theory of probability. In addition, many introductory problems are accessible to an average high school student and we would like to use this opportunity to introduce several such problems to the reader.

1. Form a square from $m \times n$-rectangles, where m and n are integers.
2. A designer gets an order to manufacture two different frames for the square window. Figure 1 shows the suggested designs. Can all the panes in either design in Figure 1 be similar rectangles ?
3. Is it possible to dissect a square into three similar rectangles, no two of which are the same ?

Readers may connect the name of the city with the famous problem about the seven bridges of Königsberg (`http://en.wikipedia.org/wiki/Seven_Bridges_of_K\"onigsberg`).

FIGURE 1: The two suggested window frame designs.

4. Is it possible to dissect a square into five squares ?
5. Each drawer's door of a cabinet is in the shape of a square (see Figure 2). Is the cabinet necessarily in the shape of a square ?

FIGURE 2: The drawers' design.

The next set of problems (problems 6–12) require more inquisitive brain work and are hard even for an A+ student. The main part of the project starts with formulation of the following result :

Theorem 1 (Freiling–Laczkovich–Rinne–Szekeres (1994)) *For $r > 0$ the following three conditions are equivalent :*

1. *A square can be tiled by rectangles with side ratio r ;*
2. *For some positive rational numbers c_i the following equality holds*

$$c_1 r + \cfrac{1}{c_2 r + \cfrac{1}{c_3 r + \cdots + \cfrac{1}{c_n r}}} = 1;$$

3. *The number r is a root of a nonzero polynomial with integer coefficients, such that all complex roots of the polynomial have positive real parts.*

Problems 13–30 constitute the core part of the project. Participants are asked to prove implications of Theorem 1 and a number of auxiliary results. To work on these tasks, one must apply some knowledge of polynomials, complex variables and mathematical theory of electrical circuits. Nine teams (sixteen students) were working on the project and the maximal advance was achieved by a team from Taiwan. A single unsolved problem from this section was problem 23 that asked to prove implication $1 \implies 3$ in Theorem 1. The following question was included even though it is an open problem.

Problem 1 *When can a cube be dissected into parallelepipeds similar to a given one?*

One of the most significant consequences of the conference is that the work on unanswered problems continues after the conference is over. In addition to continuing to investigate those problems that were not answered during the conference, the mentors also suggest that the participants look for elementary proofs of their recently proved theorems.

If you would like to try out some of these problems as well as problems associated with other projects, the materials of the 26th Summer Conference Tournament of Towns can be found at `http://www.turgor.ru/lktg/2014/`.