SOLUTIONS

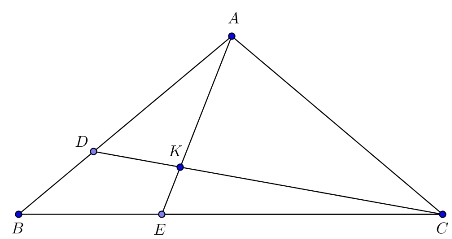
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3801. *Proposed by George Apostolopoulos.*

Triangle ABC is isosceles with $AB = AC$ and $\angle A = 100^\circ$. Let D be the point on AB such that $\angle BCD = 10^\circ$ and let E be the point on BC such that $EC = AC$. Determine the point K on CD such that triangles KAD and KCE have equal areas.

Solved by AN-anduud Problem Solving Group ; Š. Arslanagić ; R. Barbara ; R. Barroso Campos ; M. Bataille ; C. Curtis ; O. Geupel ; N. Hodžić ; D. Jonsson ; V. Konečný ; O. Kouba ; K.W. Lau ; S. Malikić ; M. Modak ; C. Sánchez-Rubio ; Skidmore College Problem Group ; D. Smith ; N. Stanciu and T. Zvonaru ; E. Swylan ; D. Văcaru ; and the proposer. We present two solutions.

Solution 1, by Oliver Geupel.



As the point K moves along the segment CD from C to D, the value of $[KCE] - [KAD]$ is strictly increasing. Hence, there is a unique location of point K such that $[KAD] = [KCE]$. Denoting the intersection of the lines AE and CD by K, we prove that K has the required property.

Using the Law of Sines in triangle ACD, we get $\dfrac{AD}{\sin 30^\circ} = \dfrac{AC}{\sin 50^\circ}$. Hence

$$AD = \frac{AC}{2\sin 50^\circ}$$

and

$$BD = AB - AD = AB \cdot \frac{2\sin 50^\circ - 1}{2\sin 50^\circ}.$$

Also, using the Law of Sines in triangle ABC, we have

$$BE = BC - CE = AB \cdot 2\sin 50^\circ - AB.$$

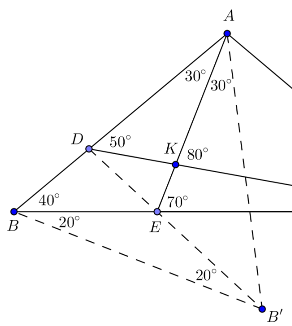
Thus,

$$\frac{BE}{BD} = 2\sin 50^\circ = \frac{BC}{AB}.$$

Therefore, the lines AC and DE are parallel. Hence the triangles ACK and EDK are homothetic. We conclude
$$\frac{AK}{EK} = \frac{CK}{DK}$$
and
$$\frac{[KAD]}{[KCE]} = \frac{AK \cdot DK}{CK \cdot EK} = 1.$$
This completes the proof.

Solution 2, by Cristóbal Sánchez-Rubio.

Let B' be the symmetric point of B with respect to the AE-axis. Then, using the given information, we can establish the angle measures as shown in the figure :



We show that the desired point K is the meeting point of the lines AE and CD.

It is easy to see that the triangle ABB' is equilateral, the angles $\angle EBB'$ and $\angle EB'B$ are both of $20°$ and $\angle BEB' = 140°$. Therefore $\angle DEB = 40^o$ and the line DE is parallel to AC.

So the triangles DEK and AKC are similar and we have :
$$\frac{KE}{KA} = \frac{KD}{KC} \iff KE \cdot KC = KA \cdot KD$$
Since $\angle AKD = \angle EKC$, the areas of KAD and KCE are the same, which completes the proof.

Editor's Comment. Bataille noticed that this is problem 983, proposed by the same author in *The College Mathematics Journal*, Vol. 43, No 4, September 2012.

3802. *Proposed by Marcel Chiriţă.*

Solve the following system
$$\sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} = 15$$
$$3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} = 3^{30}$$
for $x, y, z \in \mathbb{R}$.

Solved by AN-Anduud Problem Solving Group ; G. Apostolopoulos ; Š. Arslanagić ; M. Bataille ; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; R. Boukharfane ; M. Coiculescu ; C. Curtis ; J. L. Díaz-Barrero ; C. R. Diminnie ; R. Hess ; N. Hodžić ; S. Malikić ; P. Perfetti ; C. R. Pranesachar ; D. Smith ; D. Văcaru ; and the proposer. We present the solution by Chip Curtis.

For simplicity, let $u = \sqrt{2x+1}$, $v = \sqrt{3y+1}$, $w = \sqrt{4z+1}$ and then $p = u^2 - 1 + v$, $q = v^2 - 1 + w$ and $r = w^2 - 1 + u$. Then by AM-GM we have :

$$\begin{aligned} 3^{30} &= 3^p + 3^q + 3^r \\ &\geq 3 \cdot \sqrt[3]{3^p 3^q 3^r} \\ &= 3 \cdot 3^{\frac{p+q+r}{3}} \end{aligned}$$

with equality if and only if $p = q = r$. On the other hand, by Cauchy-Schwartz

$$\begin{aligned} p+q+r &= (u+v+w) - 3 + 3(u^2+v^2+w^2) \\ &= 12 + (u^2+v^2+w^2) \\ &\geq 12 + \frac{(u+v+w)^2}{3} \\ &= 87 \end{aligned}$$

with equality if and only if $u = v = w$.

Hence for any triple (u, v, w) with $u + v + w = 15$, we have

$$3^{u^2-1+v} + 3^{v^2-1+w} + 3^{w^2-1+u} \geq 3^{30}$$

with equality if and only if $u = v = w$. Since their sum is 15, this implies that $u = v = w = 5$ and hence

$$x = 12, y = 8, z = 6.$$

3803. *Proposed by José Luis Díaz-Barrero.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+ab} + \sqrt{c^2+bc} \leq \sqrt{2}(a+b+c)\,.$$

Solved by A. Alt ; AN-anduud Problem Solving Group ; G, Apostolopoulos ; Š. Arslanagić ; D. Bailey, E. Campbell, and C. Diminnie ; M. Bataille ; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; R. Boukharfane ; C. Curtis ; J. G. Heuver ; N. Hodžić ; T. Karamfilova ; O. Kouba ; K. Lau ; S. Malikić (2 solutions) ; D. E. Manes ; P. McCartney ; C. Mortici ; S. Muralidharan ; P. Perfetti ; C.M. Quang ; Skidmore College Problem Group ; D. Smith ; G. Tsapakadis ; D. Văcaru (2 solutions) ; H. Wang and J. Wojdylo ; P. Y. Woo ; and the proposer. We present three solutions.

Solution 1, by AN-anduud Problem Solving Group.

By the AM-GM inequality, we have :

$$\begin{aligned}\sum \sqrt{a^2+ca} &= \frac{1}{\sqrt{2}}\sum\sqrt{2a(a+c)}\\ &\le \frac{1}{\sqrt{2}}\sum\frac{2a+(a+c)}{2}\\ &= \frac{1}{2\sqrt{2}}\sum(3a+c)=\sqrt{2}(a+b+c).\end{aligned}$$

Clearly, equality holds if and only if $a=b=c$.

Solution 2, by Kee-Wai Lau.

We have

$$\sqrt{a^2+ca}=\frac{1}{2\sqrt{2}}(3a+c-(\sqrt{2a}-\sqrt{a+c})^2)\le\frac{3a+c}{2\sqrt{2}}.$$

Similarly,

$$\sqrt{b^2+ab}\le\frac{3b+a}{2\sqrt{2}}\quad\text{and}\quad\sqrt{c^2+bc}\le\frac{3c+b}{2\sqrt{2}}.$$

The result follows by adding up the three inequalities.

Solution 3, by D. M. Bătineţu-Giurgiu, Neculai Stanciu and Titu Zvonaru.

By the Cauchy-Schwarz inequality, we have :

$$(\sqrt{a}\cdot\sqrt{a+c}+\sqrt{b}\cdot\sqrt{b+a}+\sqrt{c}\cdot\sqrt{c+b})^2\le(a+b+c)(a+c+b+a+b+c)$$

or

$$\sqrt{a^2+ca}+\sqrt{b^2+ab}+\sqrt{c^2+bc}\le\sqrt{2}(a+b+c).$$

Editor's Comment. D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru proved the following generalization : let $m,n\in\mathbb{N}$ with $m\le n$, $b_j\in[0,\infty)$, $j=1,2,\dots,m$, $x_k\in(0,\infty)$, $k=1,2,\dots,n$. Let $B_m=\sum_{j=1}^m b_j$, $X_n=\sum_{k=1}^n x_k$ and $X_{k,m}=\sum_{i=0}^{m-1} b_i x_k x_{k+i}$ where the indices are taken modulo n and $k=1,2,\dots,n$. Then

$$\sum_{k=1}^n\sqrt{X_{k,m}}\le\sqrt{B_m}\cdot X_n.$$

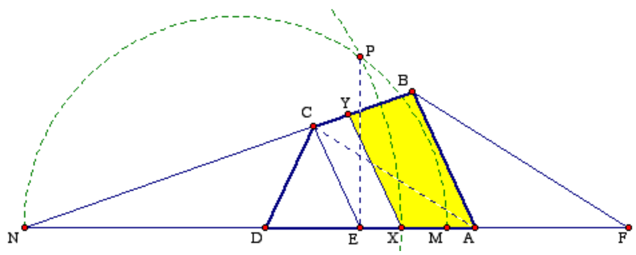
The inequality in the problem is the special case when $m=n=3$, $b_1=2$, $b_2=b_3=1$ and $(x_1,x_2,x_3)=(a,b,c)$.

3804. *Proposed by Václav Konečný.*

Let $ABCD$ be a convex quadrilateral. Construct, using only compass and straightedge, the line parallel to one side of the quadrilateral which bisects its area.

Solved by O. Geupel, C. Sánchez-Rubio, E. Swylan, P. Woo, and the proposer. We present the partial solution by Peter Woo, modified and completed by the editor.

We shall use square brackets to denote area. Label the given quadrilateral $ABCD$ so that D is further from AB than C is. Let E be the point of the side AD for which $CE||AB$, and F be the point on the line AD for which $BF||CA$. Note that the points lie in the order D, E, A, F and, because $[ABC] = [AFC]$, the area of triangle FCD equals the area of the given quadrilateral $ABCD$. Our goal is to construct the bisecting line XY parallel to AB with X on AD and Y falling on either BC or CD.



Construct M to be the midpoint of FD. Then $[MCD] = \frac{1}{2}[FCD] = \frac{1}{2}[ABCD]$. If M coincides with E, then MC is the bisecting line parallel to AB (so that $X = M$ and $Y = C$). Should E lie between D and M, then XY will necessarily lie between EC and AB (with Y on the segment BC) ; otherwise, EC will lie between XY and AB (and Y will lie on CD).

Consider first the case where E lies between M and D. Should the lines BC and AD meet in a point N, then we construct X so that NX is the geometric mean of NM and NE ; that is, $NX^2 = NM \cdot NE$. For the usual Euclidean construction when E lies between N and M as in the figure, let P be either point where the perpendicular to AD through E meets the circle whose diameter is MN ; then X is the point between M and E where the circle with centre N and radius NP intersects AD. (When N is on the other side so that M lies between E and N, use the same construction reversing the roles of E and M.) Let the line through X parallel to AB meet CB at Y. Because

$$\frac{[XYN]}{[ECN]} = \frac{NX^2}{NE^2} = \frac{NM}{NE} = \frac{[MCN]}{[ECN]},$$

we have $[XYN] = [MCN]$, whence

$$[XYCD] = [MCD] = \frac{1}{2}[ABCD],$$

as desired. This argument breaks down when the sides BC and AD are parallel (so that N is at infinity). A straightforward continuity argument places X at the midpoint of the segment ME ; alternatively, one can argue directly (using the parallelograms $AFBC$ and $EABC$) to show that when X is the midpoint of ME, the base DM of triangle CDM is twice the length of the base XA of parallelogram $XABY$ while these two polygons have the same altitude.

It remains to construct the point X in the case where M lies between D and E. Here we want $[XYD] = [MCD]$, so that now DX is the geometric mean of DE

and DM ; that is, $DX^2 = DE \cdot DM$. The proof of the claim proceeds as before with D in the role of N :

$$\frac{[XYD]}{[ECD]} = \frac{DX^2}{DE^2} = \frac{DM}{DE} = \frac{[MCD]}{[ECD]},$$

whence $[XYD] = [MCD]$, as desired.

3805. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with incentre I. Let A' be on ray IA beyond A such that $A'A = BC$. Let B' and C' be similarly defined, such that $B'B = CA$ and $C'C = AB$. Prove that

$$\frac{[A'B'C']}{[ABC]} \geq (1+\sqrt{3})^2,$$

where $[\cdot]$ denotes the area.

Solved by M. Bataille ; N. Hodžic ; O. Kouba ; S. Malikić ; M. Modak ; C. R. Pranesacher ; and the proposer. We present the solution of Madhav Modak, modified by the editor.

We note that

$$[A'B'C'] = [A'IB'] + [B'IC'] + [C'IA'].$$

Since $\angle AIB = 180^\circ - \frac{1}{2}(A+B) = 90^\circ + \frac{1}{2}C$, we have with usual notation,

$$\begin{aligned}[A'IB'] =& \frac{1}{2}A'I \cdot B'I \sin(90^\circ + \frac{1}{2}C)\\ =& \frac{1}{2}(a+AI)(b+BI)\cos(C/2)\\ =& \frac{1}{2}(ab + b \cdot AI + a \cdot BI + AI \cdot BI)\cos(C/2). \qquad (1)\end{aligned}$$

The Law of Sines for $\triangle AIB$ gives $AI/\sin(B/2) = c/\cos(C/2)$, so that

$$\frac{1}{2}b \cdot AI\cos(C/2) = \frac{1}{2}bc\sin A \cdot \frac{\sin(B/2)}{\sin A} = [ABC] \cdot \frac{\sin(B/2)}{\sin A}.$$

Similarly,

$$\frac{1}{2}a \cdot BI\cos(C/2) = [ABC] \cdot \frac{\sin(A/2)}{\sin B}.$$

Hence (1) can be written as :

$$[A'IB'] = [ABC] \cdot \frac{\cos(C/2)}{\sin C} + [ABC] \cdot \frac{\sin(B/2)}{\sin A} + [ABC] \cdot \frac{\sin(A/2)}{\sin B} + [AIB].$$

We have similar expressions for $[B'IC']$ and $[C'IA']$. Adding gives

$$\begin{aligned}[A'B'C'] =& [A'IB'] + [B'IC'] + [C'IA']\\ =& [ABC] \cdot (E_1 + E_2 + 1),\end{aligned}$$

where

$$E_1 = \frac{\cos(A/2)}{\sin A} + \frac{\cos(B/2)}{\sin B} + \frac{\cos(C/2)}{\sin C},$$
$$E_2 = \frac{\sin(B/2)+\sin(C/2)}{\sin A} + \frac{\sin(C/2)+\sin(A/2)}{\sin B} + \frac{\sin(A/2)+\sin(B/2)}{\sin C}.$$

Hence

$$\frac{[A'B'C']}{[ABC]} = E_1 + E_2 + 1. \tag{2}$$

We now prove that $E_1 \geq 3$ and $E_2 \geq 2\sqrt{3}$, which will prove the claim.

- First, for E_1 we have

$$E_1 = \frac{1}{2\sin(A/2)} + \frac{1}{2\sin(B/2)} + \frac{1}{2\sin(C/2)} = \frac{1}{2}\sum_{\text{cyclic}} \csc(A/2).$$

The convexity of $f(x) = \csc(x/2)$ on $(0, \pi)$ implies that

$$E_1 = \frac{1}{2}[f(A)+f(B)+f(C)] \geq \frac{3}{2}\cdot f[(A+B+C)/3] = \frac{3}{2}\cdot\csc(\pi/6) = 3.$$

- Next, by the *AM-GM* inequality,

$$\begin{aligned} E_2 &\geq 6\left[\frac{\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)}{\sin^2 A\sin^2 B\sin^2 C}\right]^{1/6} \\ &= 6\left[\frac{1}{64\cos^2(A/2)\cos^2(B/2)\cos^2(C/2)}\right]^{1/6} \\ &= 3\left[\frac{1}{\cos(A/2)\cos(B/2)\cos(C/2)}\right]^{1/3}. \end{aligned}$$

Another application of the *AM-GM* inequality gives

$$\cos(A/2)\cos(B/2)\cos(C/2) \leq \frac{1}{27}\left[\sum_{\text{cyclic}}\cos(A/2)\right]^3,$$

and the concavity of the function $g(x) = \cos(x/2)$ on $(0, \pi)$ gives

$$\sum_{\text{cyclic}}\cos(\frac{A}{2}) = g(A)+g(B)+g(C) \leq 3\cdot g[(A+B+C)/3)] = 3\cdot\cos(\pi/6) = \frac{3\sqrt{3}}{2}.$$

Hence,

$$\cos(A/2)\cos(B/2)\cos(C/2) \leq \frac{3\sqrt{3}}{8},$$

implying that $E_2 \geq 2\sqrt{3}$.

3806. *Proposed by Michel Bataille.*

Let triangle ABC with angles $\alpha, \beta, \gamma \neq 90°$ be inscribed in a circle with centre O and radius R, and let U, V, W be the centres of the hyperbolas with parameter R, focus O and associated directrices BC, CA, AB, respectively. Prove that

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

where $[\cdot]$ denotes the area.

Solved by C.R. Pranesachar ; D. Văcaru, and the proposer. We present the solution by C.R. Pranesachar.

Because the letters a, b, c are reserved for the sides of the given triangle ABC, we shall let

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$$

be the equation of the hyperbola whose centre is U, eccentricity is e_1, and directrix (corresponding to its focus O) is BC. If OU intersects the directrix BC at D, then $O = (a_1e_1, 0)$, $D = \left(\frac{a_1}{e_1}, 0\right)$, and

$$OD = OU - DU = \frac{a_1\left(e_1^2 - 1\right)}{e_1} = R\cos\alpha. \tag{1}$$

The parameter of the hyperbola (which equals half the length of its latus rectum) equals $R = \frac{b_1^2}{a_1}$, or (because $b_1^2 = a_1^2\left(e_1^2 - 1\right)$),

$$a_1\left(e_1^2 - 1\right) = R. \tag{2}$$

From (1) and (2) we have $e_1 = \sec\alpha$ and $a_1 = R\cot^2\alpha$, so that

$$OU = a_1e_1 = R\frac{\cos\alpha}{\sin^2\alpha}.$$

Similarly,

$$OV = R\frac{\cos\beta}{\sin^2\beta} \quad \text{and} \quad OW = R\frac{\cos\gamma}{\sin^2\gamma}.$$

But $\angle UOV = \alpha + \beta$ and $\sin(\alpha+\beta) = \sin\gamma$, so that $[OUV] = \frac{1}{2}OU \cdot OV \sin\gamma$, etc. Consequently,

$$[UVW] = [OUV] + [OVW] + [OWU] = \sum_{cyclic} \frac{R^2\cos\alpha\cos\beta}{2\sin^2\alpha\sin^2\beta}\sin\gamma.$$

Letting $F = [ABC]$ and using the formulae $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}, \sin\alpha = \frac{2F}{bc}$, etc., one has

$$[UVW] = \frac{R^2}{64F^3}\sum_{cyclic} c^2(b^2+c^2-a^2)(c^2+a^2-b^2).$$

Since $F = [ABC] = \frac{abc}{4R}$, we have after simplification,

$$[UVW]\times[ABC] = \frac{R^4}{4a^2b^2c^2}(a^6+b^6+c^6-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)+6a^2b^2c^2).$$

But

$$\begin{aligned}\cos^2\alpha+\cos^2\beta+\cos^2\gamma &= \\ &= \sum_{cyclic}\frac{(b^2+c^2-a^a)^2}{4b^2c^2} = \frac{\sum a^2(b^2+c^2-a^a)^2}{4a^2b^2c^2} \\ &= \frac{a^6+b^6+c^6-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)+6a^2b^2c^2}{4a^2b^2c^2}.\end{aligned}$$

Thus,

$$[UVW]\times[ABC] = R^4(\cos^2\alpha+\cos^2\beta+\cos^2\gamma),$$

as desired.

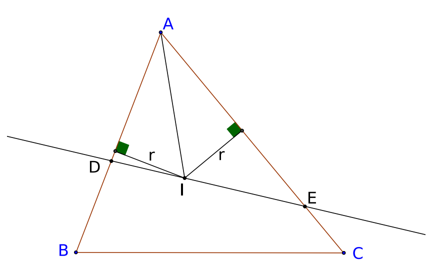
3807. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

where r denotes the inradius of ABC.

Solved by A. Alt ; M. Amengual Covas ; AN-anduud Problem Solving Group ; Š. Arslanagić ; R. Barroso Campos ; M. Bataille ; R. Barbara ; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru ; C. Curtis ; P. De ; O. Geupel ; D. Jonsson ; O. Kouba ; S. Malikić ; M. Modak ; C.R. Pranesachar ; C. M. Quang ; C. Sánchez-Rubio ; E. Swylan ; G. Tsapakidis ; D. Văcaru ; P.Y. Woo ; and the proposer. We present the solution by Oliver Geupel.



Denoting area by $[\cdot]$, we have

$$AD\cdot AE \geq AD\cdot AE\sin\angle A = 2[ADE] = 2[ADI]+2[AEI] = AD\cdot r + AE\cdot r.$$

Hence

$$r \leq \frac{AD \cdot AE}{AD + AE}$$

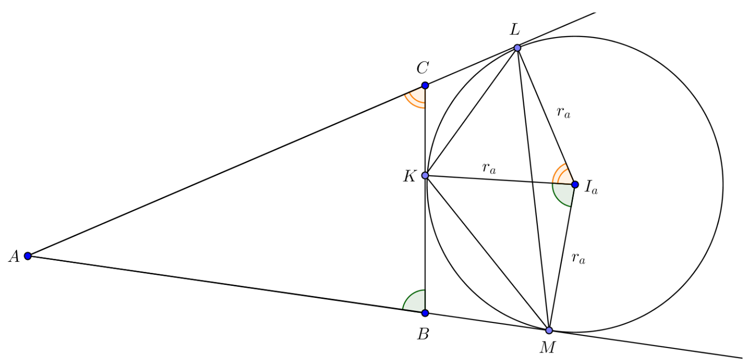
and the result follows immediately. The equality holds if and only if $\angle A$ is a right angle.

3808. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with area Δ ; circumradius R ; exradii r_a, r_b, r_c ; and excenters I_a, I_b, I_c. The excircle with centre I_a touches the sides of ABC at K, L, and M. Let Δ_1 represent the area of triangle KLM and let Δ_2 and Δ_3 be similarly defined. Prove that

$$\frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{r_a + r_b + r_c}{2R}.$$

Solved by A. Alt ; M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; P. De ; O. Geupel ; J. Heuver ; O. Kouba ; S. Malikić ; C.R. Pranesachar ; C. Sánchez-Rubio ; G. Tsapakidis ; D. Văcaru ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present a composite of similar solutions by Arkady Alt, Miguel Amengual Covas, and Oliver Geupel.



We use the common notation

$$a = BC, \quad b = CA, \quad c = AB, \quad 2s = a + b + c.$$

Since quadrilaterals MI_aKB, I_aLCK, and MI_aLA are cyclic, we have

$$\angle MI_aK = \angle B, \quad \angle KI_aL = \angle C, \quad \text{and} \quad \angle MI_aL = \angle 180^\circ - \angle A.$$

It follows that

$$\begin{aligned} \Delta_1 &= [I_aKM] + [I_aLK] - [I_aLM] \\ &= \frac{r_a^2}{2}(\sin B + \sin C - \sin(180^\circ - A)) \\ &= \frac{r_a^2}{2}(\sin B + \sin C - \sin A) \\ &= \frac{r_a^2}{2} \cdot \frac{b + c - a}{2R} = \frac{r_a}{2R} \cdot r_a(s - a) = \frac{r_a}{2R}\Delta. \end{aligned}$$

Analogously,

$$\Delta_2 = \frac{r_b}{2R}\Delta, \quad \Delta_3 = \frac{r_c}{2R}\Delta,$$

hence the result.

3809. *Proposed by Michel Bataille.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

Solved by AN-anduud Problem Solving Group ; R. Boukharfane ; C. Curtis ; P. Deiermann and H. Wang ; O. Kouba ; K. W. Lau ; P. Perfetti ; D. Smith ; and the proposer. One incorrect solution was received. We present the solution by Paolo Perfetti.

The given inequality is equivalent to

$$x^{\frac{x}{2}} x^{\frac{y}{2}} \geq \left(\sqrt{\frac{x^2+y^2}{2}}\right)^{\frac{x+y}{2}} \iff x^{\frac{2x}{x+y}} y^{\frac{2y}{x+y}} \geq \frac{x^2+y^2}{2},$$

which upon being divided by x^2 becomes

$$\frac{y^{\frac{2y}{x+y}}}{x^{\frac{2y}{x+y}}} \geq \frac{1}{2}\left(1 + \left(\frac{y}{x}\right)^2\right). \tag{1}$$

Without loss of generality, we assume that $x \leq y$. Let $t = \frac{y}{x}$. Then $t \geq 1$, $\frac{2y}{x+y} = \frac{2t}{1+t}$ and (1) becomes

$$t^{\frac{2t}{1+t}} \geq \frac{1+t^2}{2} \iff \frac{2t}{1+t}\ln t \geq \ln\left(\frac{1+t^2}{2}\right). \tag{2}$$

To prove (2), let $f(t) = \frac{2t}{1+t}\ln t - \ln\frac{1+t^2}{2}$, $t \geq 1$. Then by routine calculations, we find :

$$f'(t) = 2\left(\frac{1-t^2+(1+t^2)\ln t}{(1+t)^2(1+t^2)}\right).$$

We claim that

$$1 - t^2 + (1+t^2)\ln t \geq 0 \quad \text{for all} \quad t \geq 1. \tag{3}$$

Let $h(t) = \ln t - \dfrac{t^2-1}{1+t^2} = \ln t - 1 + \dfrac{2}{1+t^2}$. Then

$$h'(t) = \frac{1}{t} - \frac{4t}{(1+t^2)^2} = \frac{(1-t^2)^2}{t(1+t^2)^2} \geq 0,$$

so $h(t)$ is an increasing function.

Since $h(1) = 0$, we have $h(t) \ge 0$, from which (3) follows. Hence, $f'(t) \ge 0$, which implies that $f(t)$ is an increasing function. Since $f(1) = 0$, we conclude that $f(t) \ge 0$ for all $t \ge 1$, which establishes (2) and completes the proof.

3810. *Proposed by Ovidiu Furdui.*

Let $k > 0$ be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Solved by Š. Arslanagić; R. Boukharfane; C. Curtis; O. Geupel; R. I. Hess; O. Kouba; J. Ling; D. Stone and J. Hawkins; and the proposer. One incorrect solution was received, although the error was of a purely algebraic variety. We present two solutions.

Solution 1, by Oliver Geupel.

For $0 \le x \le 1$, let $f(x) = \int\limits_0^1 \left\{ \frac{x}{y} \right\} dy$. Let $y = \frac{x}{t}$, then $dy = -\frac{x}{t^2}\, dt$ and we obtain :

$$\begin{aligned}
f(x) = x \int_x^\infty \frac{\{t\}}{t^2}\, dt &= x \left(\int_x^1 \frac{dt}{t} + \lim_{n\to\infty} \sum_{\ell=1}^{n} \int_\ell^{\ell+1} \frac{t-\ell}{t^2}\, dt \right) \\
&= x \left(-\log x + \lim_{n\to\infty} \sum_{\ell=1}^{n} \left(\log(\ell+1) - \log \ell + \frac{\ell}{\ell+1} - 1 \right) \right) \\
&= -x \log x + x \cdot \lim_{n\to\infty} \left(\log(n+1) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \right) \right) \\
&= -x \log x + x(1-\gamma).
\end{aligned}$$

Hence, $f(x^k) = x^k(1-\gamma) - kx^k \log x$.

Let I be the integral to be evaluated. Then we have :

$$\begin{aligned}
I = \int_0^1 f(x^k)\, dx &= \frac{1-\gamma}{k+1} - k \int_0^1 x^k \log x\, dx \\
&= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \int_u^1 x^k \log x\, dx \\
&= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \left[\frac{1}{k+1} x^{k+1} \log x - \frac{x^{k+1}}{(k+1)^2} \right]_u^1 \\
&= \frac{1-\gamma}{k+1} + \frac{k}{(k+1)^2},
\end{aligned}$$

because $\lim_{u\to 0+} u^{k+1}\log u = 0$ by l'Hôpital's rule.

Solution 2, by Chip Curtis slightly modified by the editor.

We consider this as an integral over a two-dimensional region, and perform a change of variables. Let

$$u = \frac{x^k}{y},\ \ v = y;$$

let $\alpha = \dfrac{1}{k}$, so that we obtain $x = (uv)^\alpha$. The Jacobian is given by :

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \alpha u^{\alpha-1}v^\alpha & \alpha u^\alpha v^{\alpha-1} \\ 0 & 1 \end{vmatrix} = \alpha u^{\alpha-1}v^\alpha.$$

The image $\mathcal{S}$ of the region $\mathcal{R} = [0,1]\times[0,1]$ is

$$\begin{aligned}\mathcal{S} &= \left\{(u,v) : 0 \le v \le 1 \text{ and } 0 \le u \le \frac{1}{v}\right\} \\ &= \{(u,v) : 0 \le u \le 1 \text{ and } 0 \le v \le 1\} \cup \left\{(u,v) : 1 \le u \le \infty \text{ and } 0 \le v \le \frac{1}{u}\right\}.\end{aligned}$$

The integral then becomes :

$$\begin{aligned} I(k) &= \alpha \iint_{\mathcal{S}} \{u\}u^{\alpha-1}v^\alpha\ dvdu \\ &= \alpha\int_0^1\int_0^1 \{u\}u^{\alpha-1}v^\alpha\ dvdu + \alpha\int_1^\infty\int_0^{\frac{1}{u}} \{u\}u^{\alpha-1}v^\alpha\ dvdu \\ &= \alpha\int_0^1\int_0^1 u^\alpha v^\alpha\ dvdu + \alpha\int_1^\infty \{u\}u^{\alpha-1}\left(\int_0^{\frac{1}{u}} v^\alpha\ dv\right)\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \alpha\int_1^\infty \{u\}u^{\alpha-1}\left(\frac{1}{(\alpha+1)u^{\alpha+1}}\right)\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \frac{\alpha}{\alpha+1}\int_1^\infty \{u\}u^{-2}\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \frac{\alpha}{\alpha+1}(1-\gamma) \\ &= \frac{k}{(k+1)^2} + \frac{1-\gamma}{k+1}, \end{aligned}$$

where the integral at the end of the computation is part of the one performed in the previous solution, presented above.

Editor's comments. Two main methods of proof were utilized. One method involved using a substitution ; some did a 1D swap just to compute the inside integral, and others used a full 2D change of variables, complete with Jacobian factor. The other method featured a computation of the branches of $\{\frac{x^k}{y}\}$ in the unit square, then summing the integrals over each region. The reader may wish to prove that the limit defining the Euler-Mascheroni number γ is convergent. Two comments regarding this problem were received, both stating that the problem and the proposer's solution appear (on pages 104 and 129, respectively) in the proposer's 2013 book, *Limits, Series, and Fractional Part Integrals*, published by Springer.