PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

Submissions of proposals. *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions should be received by the editor by **1 May 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

3901. *Proposed by D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Let $A, B \in M_n(\mathbb{R})$ with $\det A = \det B \neq 0$. If $a, b \in \mathbb{R} \setminus \{0\}$, prove that

$$\det(aA + bB^{-1}) = \det(aB + bA^{-1}).$$

3902. *Proposed by Michel Bataille.*

Let ABC be a triangle with $AB = AC$ and $\angle BAC \neq 90^\circ$ and let O be its circumcentre. Let M be the midpoint of AC and let P on the circumcircle of $\triangle AOB$ be such that $MP = MA$ and $P \neq A$. The lines l and m pass through A and are perpendicular and parallel to PM, respectively. Suppose that the lines l and PC intersect at U and that the line PB intersect AC at V and m at W. Prove that U, V and W are not collinear and that l is tangent to the circumcircle of $\triangle UVW$.

3903. *Proposed by George Apostolopoulos.*

Consider a triangle ABC with an inscribed circle with centre I and radius r. Let C_A, C_B and C_C be circles internal to ABC, tangent to its sides and tangent to the inscribed circle with the corresponding radii r_A, r_B and r_C. Show that $r_A + r_B + r_C \geq r$.

3904. *Proposed by Abdilkadir Altintaş.*

Let ABC be an equilateral triangle and let D, E and F be the points on the sides AB, BC and AC, respectively, such that $AD = 2$, $AF = 1$ and $FC = 3$. If the triangle DEF has minimum possible perimeter, what is the length of AE?

3905. *Proposed by Jonathan Love.*

A sequence $\{a_n : n \geq 2\}$ is called *prime-picking* if, for each n, a_n is a prime divisor of n. A sequence $\{a_n : n \geq 2\}$ is called *spread-out* if, for each positive integer k, there is an index N such that, for $n \geq N$, the k consecutive entries $a_n, a_{n+1}, \cdots, a_{n+k-1}$ are all distinct. For example, the sequence

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

is spread-out. Does there exist a prime-picking spread-out sequence?

3906★. *Proposed by Titu Zvonaru and Neculai Stanciu.*

If $x_1, x_2, \ldots, x_n$ are positive real numbers, then prove or disprove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

for all positive integers n.

3907. *Proposed by Enes Kocabey.*

Let $ABCDEF$ be a convex hexagon such that $AB+DE = BC+EF = FA+CD$ and $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Let the midpoints of the sides AF, CD, BC and EF be M, N, K and L, respectively, and let $MN \cap KL = \{P\}$. Show that $\angle BCD = 2\angle KPN$.

3908. *Proposed by George Apostolopoulos.*

Prove that $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ for each integer $n \geq 3$.

3909. *Modified proposal of Victor Oxman, Moshe Stupel and Avi Sigler.*

Given an acute-angled triangle together with its circumcircle and orthocentre, construct, with straightedge alone, its circumcentre.

Editor's Comment. The Poncelet-Steiner Theorem (1833) states that whatever can be constructed by straightedge and compass together can be constructed by straightedge alone, given a circle and its centre ; but Steiner showed that given only the circle and a straightedge, the centre cannot be found. (This shows that the orthocentre must be given in the present problem ; it cannot be constructed with the straightedge and circumcircle !) Details can be found in texts such as A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), or on the internet by googling the Poncelet-Steiner Theorem.

3910. *Proposed by Paul Yiu.*

Two triangles ABC and $A'B'C'$ are homothetic. Show that if B' and C' are on the perpendicular bisectors of CA and AB respectively, then A' is on the perpendicular bisector of BC, and the homothetic center is a point on the Euler line of ABC.

. . . .

3901. *Proposé par D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Soient $A, B \in M_n(\mathbb{R})$ telles que $\det A = \det B \neq 0$. Si $a, b \in \mathbb{R}^*$, démontrer que

$$\det(aA + bB^{-1}) = \det(aB + bA^{-1}).$$

3902. *Proposé par Michel Bataille.*

Soit ABC un triangle tel que $AB = AC$ et $\angle BAC \neq 90°$; soit O le centre de son cercle circonscrit. Soit M le mipoint de AC et soit P sur le cercle circonscrit de $\triangle AOB$ tel que $MP = MA$ and $P \neq A$. Les lignes l et m passent par A et sont perpendiculaire et parallèle à PM respectivement. Les lignes l et PC intersectent à U, puis que la ligne PC intersecte AC à V et m à W. Démontrer que U, V et W sont non colinéaires et que l est tangente au cercle circonscrit de $\triangle UVW$.

3903. *Proposé par George Apostolopoulos.*

Considérer un triangle ABC avec cercle inscrit de centre I et rayon r. Soient CA, CB et CC les cercles internes à ABC, tangents à ses côtés et au cercle inscrit, avec rayons correspondants r_A, r_B et r_C. Démontrer que $r_A + r_B + r_C \geq r$.

3904. *Proposé par Abdilkadir Altintaş.*

Soit ABC un triangle équilatéral et soient D, E et F les points sur les côtés AB, BC et AC respectivement, tels que $AD = 2, AF = 1$ et $FC = 3$. Si le triangle DEF a le périmètre le plus petit possible, quelle est la longueur de AE ?

3905. *Proposé par Jonathan Love.*

Une suite $\{a_n : n \geq 2\}$ est dite *premier-cueillante* si, pour tout n, a_n est un diviseur premier de n. Une suite $\{a_n : n \geq 2\}$ est dite *écartée* si, pour tout entier positif k, il existe un entier N tel que, pour $n \geq N$, les k valeurs consécutives $a_n, a_{n+1}, \cdots, a_{n+k-1}$ sont distinctes. Par exemple, la suite

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

est écartée. Existe-t-il une suite à la fois premier-cieuillante et écartée ?

3906$\star$. *Proposé par Titu Zvonaru et Neculai Stanciu.*

Si $x_1, x_2, \ldots, x_n$ sont des nombres réels positifs, prouver vrai ou prouver faux que

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

quelque soit n entier positif.

3907. *Proposé par Enes Kocabey.*

Soit $ABCDEF$ un hexagone convexe tel que $AB + DE = BC + EF = FA + CD$, puis que $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Les mipoints des côtés AF, CD, BC et EF sont dénotés M, N, K et L, respectivement ; aussi, $MN \cap KL = \{P\}$. Démontrer que $\angle BCD = 2\angle KPN$.

3908. *Proposé par George Apostolopoulos.*

Démontrer que $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ pour tout entier $n \geq 3$.

3909. *Proposition modifiée de Victor Oxman, Moshe Stupel et Avi Sigler.*

Étant donné un triangle aïgu, son cercle circonscrit et son orthocentre, construire le centre du cercle circonscrit à l'aide de la règle.

Commentaire de l'éditeur. Selon le théorème Poncelet-Steiner (1883), étant donné un seul cercle et son centre, toute construction possible à l'aide de règle et compas est possible à l'aide de règle ; mais Steiner a aussi démontré qu'étant donné seulement un cercle et la règle, le centre ne peut pas être construit. (Ceci montre que l'orthocentre doit être donné dans le problème ci-haut, car il ne peut pas être construit à partir de règle et le cercle circonscrit.) Les détails se trouvent notamment dans le livre A. S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), ou aussi en googlant "Poncelet-Steiner Theorem".

3910. *Proposé par Paul Yiu.*

Deux triangles ABC et $A'B'C'$ sont homothétiques. Démontrer que si B' et C' se trouvent sur les bissectrices perpendiculaires de CA et AB respectivement, il ensuit que A' se trouve sur la bissectrice perpendiculaire de BC et que le centre homothétique est un point sur la ligne d'Euler de ABC.