THE OLYMPIAD CORNER

No. 319

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-olympiad@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 May 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.

OC161. The altitude BH dropped onto the hypotenuse AC of a right triangle ABC intersects the angle bisectors AD and CE at Q respectively P. Prove that the line passing through the midpoints of segments $[QD]$ and $[PE]$ is parallel to the line AC.

OC162. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $k, m, n \in \mathbb{N}$ we have

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

OC163. Let $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ and S be the set of all subsets of A. Find the number of functions $f : S \to B$ such that

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S\,.$$

OC164. Find all triples (m, p, q) where m is a positive integer and p, q are primes such that

$$2^m p^2 + 1 = q^5\,.$$

OC165. Let O be the circumcenter of acute ΔABC, and let H be its orthocenter. Let $AD \perp BC$, and let EF be the perpendicular bisector of AO, with D, E on the side BC. Prove that the circumcircle of ΔADE passes through the midpoint of OH.

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OC161. L'altitude BH vers l'hypotenuse AC du triangle ABC intersecte les bissectrices de AD et CE à Q et P respectivement. Démontrer que la ligne passant par les mipoints des segments $[QD]$ et $[PE]$ est parallèle à la ligne AC.

OC162. Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{R}$ telles que pour tout $k, m, n \in \mathbb{N}$, l'inégalité qui suit est valide

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

OC163. Soit $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ et S l'ensemble de tous les sous ensembles de A. Déterminer le nombre de fonctions $f : S \to B$ telles que

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ pour tout } A_1, A_2 \in S\,.$$

OC164. Déterminer tous les triplets (m, p, q) où m est un entier positif et p, q sont des nombres premiers tels que

$$2^m p^2 + 1 = q^5\,.$$

OC165. Soit O le centre du cercle circonscrit du triangle aïgu ΔABC et soit H l'orthocentre. Soit $AD \perp BC$ et soit EF la bissectrice perpendiculaire de AO, avec D et E sur le côté BC. Démontrer que le cercle circonscrit de ΔADE passe par la mipoint de OH.

OLYMPIAD SOLUTIONS

OC101. Let n, k be positive integers so that $1 < k < n-1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.

Originally question 5 from the 2011 Estonia Team Selection Test, Day 2.

No solution was received to this problem. We give the official solution from Estonian Math Competitions 2010/2011*, The Gifted and Talented Development Centre, Tartu, 2011.*

We can assume without loss of generality that $2k \leq n$, otherwise we interchange k with $n-k$.

Assume by contradiction that

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

for some prime p and some integer n. Write every number in the numerator in the form $n - i = p^{\alpha_i} s_i$ with $p \nmid s_i$, where $0 \leq i \leq k-1$.

First let us observe that we have $s_i \neq s_j$. Indeed, assume by contradiction that $s_i = s_j$ for some $i < j$. Then, as $p^{\alpha_i} s_i = n - i > n - j = p^{\alpha_j} s_j$ we get $\alpha_i \geq 1 + \alpha_j$. Therefore

$$n \geq p^{\alpha_i} s_i \geq p p^{\alpha_j} s_i = p(n-j) > p(n-k) \geq 2(n-k) \,,$$

which contradicts $2k \leq n$.

This shows that the k terms $s_0, s_1, \ldots, s_{k-1}$ at the top are pairwise distinct.

Moreover, as the numerator contains at least two consecutive integers, at least one of these is not divisible by p. Therefore, there exists some j so that $s_j = n - j > n - k \geq k$.

As the elements $s_0, s_1, \ldots, s_{k-1}$ are pairwise distinct, and at least one of them is strictly greater than k, we have

$$s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k \,.$$

Moreover, as

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

we get

$$\prod p^{\alpha_i} s_i = p^l 1 \cdot 2 \cdot \ldots \cdot k \Rightarrow s_1 \cdot .. \cdot s_k | p^l 1 \cdot 2 \cdot ... \cdot k$$

As each s_i is not divisible by p, $s_1 \cdot \ldots \cdot s_k$ is relatively prime with p^l. Therefore

$$s_1 \cdot \ldots \cdot s_k | 1 \cdot 2 \cdot \ldots \cdot k$$

But this contradicts $s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k$.

As we got a contradiction, our assumption is wrong, therefore $\binom{n}{k}$ cannot be a power of a prime.

OC102. Let $\mathbb{N}$ denote the set of all nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that

1. $0 \le f(x) \le x^2$ for all $x \in \mathbb{N}$
2. $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$ with $x > y$.

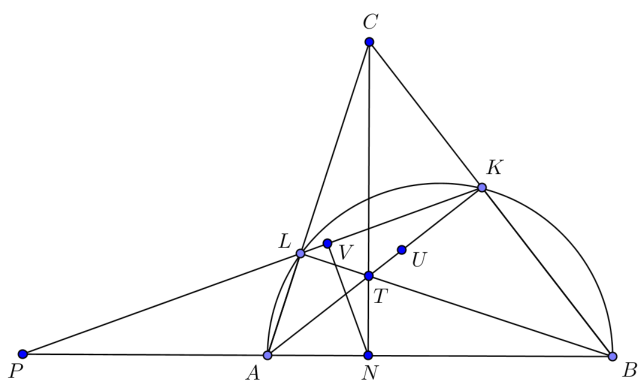
Originally question 5 from the 2011 South Africa National Olympiad.

No solution was received to this problem.

OC103. Let K and L be points on a semicircle with diameter AB. Denote the intersection of AK and BL as T and let N be the point such that N is on segment AB and line TN is perpendicular to AB. If U is the intersection of the perpendicular bisectors of AB and KL and V is a point on KL such that angles UAV and UBV are equal, then prove that NV is perpendicular to KL.

Originally question 3 from 2011 Croatia Team Selection Test, Day 2.

Solved by Š. Arslanagić; and D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru. We give the solution of the latter group.



Denote the intersection of AL and BK by C. Then AK and BL are altitudes in the triangle ABC, and therefore T is the orthocenter in this triangle. This implies that C, T, N are collinear.

The condition $\angle UAV = \angle UBV$ means that the quadrilateral $ABUV$ is cyclic.

Let M be the midpoint of AB. We will prove that $MUVN$ is cyclic, which shows that $NV \perp KL$.

If $KL \parallel AB$, then ABC is isosceles, in which case M coincides with N and U coincides with V, so $NV \perp KL$.

Otherwise, let P be the intersection of AB and KL. By symmetry, we can assume that A is between P and B.

As usual we denote by A, B, C respectively a, b, c the angles respectively the sides in the triangle ABC.

By applying the Menelaus theorem in triangle ABC with transversal $P-L-K$ we obtain

$$\frac{PA}{PB}\frac{KB}{KC}\frac{LC}{LA}=1 \Leftrightarrow \frac{PA}{c+PA}c\cos(B)b\cos(C)\frac{a\cos(C)}{c\cos(A)}=1 \Leftrightarrow \frac{PA}{c+PA}=\frac{b\cos(A)}{a\cos(B)}.$$

Therefore

$$PA=\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\,.$$

As $ABUV$ is cyclic, we have

$$PU\cdot PV=PA\cdot PB\,.$$

To complete the proof, we need to show that

$$PU\cdot PV=PM\cdot PN\,.$$

We have

$$\begin{aligned}
PU\cdot PV&=PM\cdot PN \Leftrightarrow\\
PA\cdot PB&=PM\cdot PN \Leftrightarrow\\
PA\cdot(c+PA)&=(PA+b\cos(A))(PA+\frac{c}{2}) \Leftrightarrow\\
PA^2+cPA&=PA^2+bPA\cos(A)+PA\frac{c}{2}+\frac{c}{2}b\cos(A) \Leftrightarrow\\
PA\frac{c}{2}&=bPA\cos(A)+\frac{c}{2}b\cos(A) \Leftrightarrow\\
\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\frac{c}{2}&=b\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\cos(A)+\frac{bc\cos(A)}{2} \Leftrightarrow\\
\frac{c}{2(a\cos(B)-b\ cos(A))}&=\frac{2b\cos(A)}{2(a\cos(B)-b\ cos(A))}+\frac{a\cos(B)-b\ cos(A)}{2(a\cos(B)-b\ cos(A))} \Leftrightarrow\\
c&=2b\cos(A)+a\cos(B)-b\ cos(A) \Leftrightarrow\\
c&=b\cos(A)+a\cos(B)
\end{aligned}$$

which is true.

OC104. Given a triangle ABC, let D be the midpoint of the side AC and let M be the point on the segment BD so that $BM : MD = 1 : 2$. The rays AM and CM intersect the sides BC and AB at E respectively F. We know that $AM \perp CM$. Prove that the quadrangle $AFED$ is cyclic if and only if the median from A in ΔABC meets the line EF at a point situated on the circumcircle of ΔABC.

Originally question 3 from the 2011 Romania Team Selection Test, Day 4.

Solved by D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; and O. Geupel. We give the solution of Oliver Geupel.

Let $a = BC$, $b = CA$, and $c = AB$.

By Apollonius' theorem, the squared length of the triangle median BD is

$$BD^2 = \frac{2(a^2 + c^2) - b^2}{4}\,.$$

By the assertions $BM : MD = 1 : 2$ and $AM \perp CM$, we have $\frac{2}{3}BD = DM = \frac{b}{2}$. It follows that

$$13b^2 = 8(a^2 + c^2). \tag{1}$$

Using barycentric coordinates, we have

$$\begin{aligned} M &= \frac{1}{6}A + \frac{2}{3}B + \frac{1}{6}C \\ E &= \frac{4}{5}B + \frac{1}{5}C \\ F &= \frac{1}{5}A + \frac{4}{5}B. \end{aligned}$$

Therefore $BE = \frac{a}{5}$, $BF = \frac{c}{5}$, $EF = \frac{b}{5}$, and $EF \parallel AD$.

We show that the quadrilateral $AFED$ is cyclic if and only if

$$b^2 = 2c^2 \quad \text{and} \quad a^2 = \frac{9}{4}c^2. \tag{2}$$

By the law of cosines in $\triangle CDE$ we have

$$\begin{aligned} DE^2 &= CE^2 + CD^2 - 2CE \cdot CD \cos C \\ &= \frac{16}{25}a^2 + \frac{1}{4}b^2 - 2 \cdot \frac{4}{5}a \cdot \frac{1}{2}b \cdot \frac{a^2 + b^2 - c^2}{2ab} && (3) \\ &= \frac{6}{25}a^2 - \frac{3}{20}b^2 + \frac{2}{5}c^2. && (4) \end{aligned}$$

The trapezoid $AFED$ with $EF \neq AD$ is cyclic if and only if it is isosceles, i.e. $DE = \frac{4}{5}c$. By (3), this is equivalent to

$$8a^2 = 5b^2 + 8c^2\,.$$

In view of (1), it simplifies to (2). Consequently, quadrilateral $AFED$ is cyclic if and only if (2).

It suffices to show that the median from A meets the line EF at a point situated on the circumcircle of $\triangle ABC$ if and only if (2) holds.

Let N be the midpoint of BC and let P be the intersection of lines EF and AN.

By Apollonius' theorem the squared length of the median AN is

$$AN^2 = \frac{2(b^2+c^2)-a^2}{4}\,.$$

By Menelaus' theorem applied to the line EF in $\triangle ABN$ we have

$$\frac{PN}{PN+AN} = \frac{BF}{AF}\cdot\frac{NE}{BE} = \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8},$$

whence $PN = \frac{3}{5}AN$.

Observe that the point P lies on the circumcircle of triangle ABC if and only if $BN \cdot CN = AN \cdot PN$. This is successively equivalent to

$$\frac{a^2}{4} = \frac{3}{20}(2(b^2+c^2)-a^2)$$

and to $4a^2 = 3(b^2+c^2)$. In view of (1), this simplifies to (2).

This completes the proof.

OC105. Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer $a, 1 < a < \frac{n}{k}+1$, such that $n \mid a^2 - a$.

Originally question 2 from 2011 China Team Selection, Quiz 3, Day 1.

Solved by Oliver Geupel whose solution is presented below.

Let $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ be the prime factorization of n.

Then, the condition $n \mid a(a-1)$ holds if and only if there are numbers $b_1, b_2, \ldots, b_k \in \{0,1\}$ such that

$$\begin{aligned} a &\equiv b_1 \pmod{p_1^{e_1}} \\ a &\equiv b_2 \pmod{p_2^{e_2}} \\ &\ \cdots \\ a &\equiv b_k \pmod{p_k^{e_k}}\,. \end{aligned} \qquad (5)$$

For $i, j \in \{1, 2, \ldots, k\}$, define

$$b_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases}$$

For $j = 1, \ldots, k$, let $\mathcal{S}_j$ denote the following system of simultaneous congruences :

$$\begin{aligned} a &\equiv b_{1j} \pmod{p_1^{e_1}} \\ a &\equiv b_{2j} \pmod{p_2^{e_2}} \\ &\ldots \\ a &\equiv b_{kj} \pmod{p_k^{e_k}}\,. \end{aligned}$$

By the Chinese remainder theorem, there are k numbers $a_1, \ldots, a_k$, such that for $j = 1, \ldots, k$ it holds $2 \le a_j \le n$, and $a = a_j$ solves the simultaneous congruences $\mathcal{S}_j$.

The k closed intervals

$$\begin{aligned} I_1 &= \left[2,\ \frac{n}{k}\right], \\ I_2 &= \left[\frac{n}{k} + 1,\ 2 \cdot \frac{n}{k}\right], \\ I_3 &= \left[2 \cdot \frac{n}{k} + 1,\ 3 \cdot \frac{n}{k}\right], \\ &\ldots, \\ I_k &= \left[(k-1) \cdot \frac{n}{k} + 1,\ n\right] \end{aligned}$$

constitute a disjoint partition of the interval $[2, n]$.

If at least one of the numbers a_j, $1 \le j \le k$ belongs to I_1, then this number has the required property.

Otherwise, there are two distinct numbers a', a'' among $a_1, \ldots, a_k$ that belong to the same interval I_j by the Pigeonhole principle.

But then, one of the numbers $|a' - a''|$ and $|a' - a''| + 1$ satisfies a system of simultaneous congruences of the form (5) and belongs to I_1.

This completes the proof.