# OLYMPIAD SOLUTIONS

**OC101**. Let $n, k$ be positive integers so that $1 < k < n-1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.

*Originally question 5 from the 2011 Estonia Team Selection Test, Day 2.*

*No solution was received to this problem. We give the official solution from* Estonian Math Competitions 2010/2011*, The Gifted and Talented Development Centre, Tartu, 2011.*

We can assume without loss of generality that $2k \leq n$, otherwise we interchange $k$ with $n-k$.

Assume by contradiction that

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

for some prime $p$ and some integer $n$. Write every number in the numerator in the form $n - i = p^{\alpha_i} s_i$ with $p \nmid s_i$, where $0 \leq i \leq k-1$.

First let us observe that we have $s_i \neq s_j$. Indeed, assume by contradiction that $s_i = s_j$ for some $i < j$. Then, as $p^{\alpha_i} s_i = n - i > n - j = p^{\alpha_j} s_j$ we get $\alpha_i \geq 1 + \alpha_j$. Therefore

$$n \geq p^{\alpha_i} s_i \geq p p^{\alpha_j} s_i = p(n-j) > p(n-k) \geq 2(n-k) \,,$$

which contradicts $2k \leq n$.

This shows that the $k$ terms $s_0, s_1, \ldots, s_{k-1}$ at the top are pairwise distinct.

Moreover, as the numerator contains at least two consecutive integers, at least one of these is not divisible by $p$. Therefore, there exists some $j$ so that $s_j = n - j > n - k \geq k$.

As the elements $s_0, s_1, \ldots, s_{k-1}$ are pairwise distinct, and at least one of them is strictly greater than $k$, we have

$$s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k \,.$$

Moreover, as

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

we get

$$\prod p^{\alpha_i} s_i = p^l 1 \cdot 2 \cdot \ldots \cdot k \Rightarrow s_1 \cdot .. \cdot s_k | p^l 1 \cdot 2 \cdot ... \cdot k$$

As each $s_i$ is not divisible by $p$, $s_1 \cdot \ldots \cdot s_k$ is relatively prime with $p^l$. Therefore

$$s_1 \cdot \ldots \cdot s_k | 1 \cdot 2 \cdot \ldots \cdot k$$

But this contradicts $s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k$.

As we got a contradiction, our assumption is wrong, therefore $\binom{n}{k}$ cannot be a power of a prime.

**OC102**. Let $\mathbb{N}$ denote the set of all nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that

1. $0 \leq f(x) \leq x^2$ for all $x \in \mathbb{N}$
2. $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$ with $x > y$.

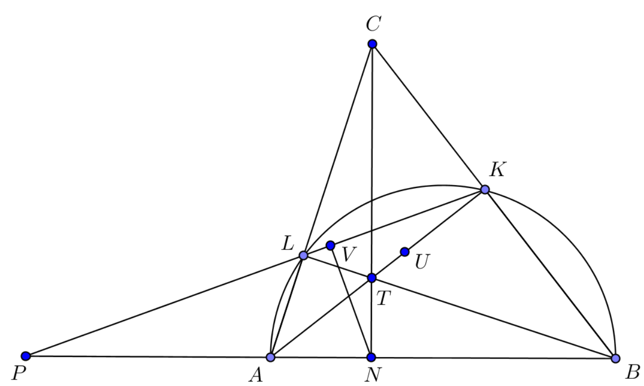
*Originally question 5 from the 2011 South Africa National Olympiad.*

*No solution was received to this problem.*

**OC103**. Let $K$ and $L$ be points on a semicircle with diameter $AB$. Denote the intersection of $AK$ and $BL$ as $T$ and let $N$ be the point such that $N$ is on segment $AB$ and line $TN$ is perpendicular to $AB$. If $U$ is the intersection of the perpendicular bisectors of $AB$ and $KL$ and $V$ is a point on $KL$ such that angles $UAV$ and $UBV$ are equal, then prove that $NV$ is perpendicular to $KL$.

*Originally question 3 from 2011 Croatia Team Selection Test, Day 2.*

*Solved by Š. Arslanagić; and D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru. We give the solution of the latter group.*




Denote the intersection of $AL$ and $BK$ by $C$. Then $AK$ and $BL$ are altitudes in the triangle $ABC$, and therefore $T$ is the orthocenter in this triangle. This implies that $C, T, N$ are collinear.

The condition $\angle UAV = \angle UBV$ means that the quadrilateral $ABUV$ is cyclic.

Let $M$ be the midpoint of $AB$. We will prove that $MUVN$ is cyclic, which shows that $NV \perp KL$.

If $KL \parallel AB$, then $ABC$ is isosceles, in which case $M$ coincides with $N$ and $U$ coincides with $V$, so $NV \perp KL$.

Otherwise, let $P$ be the intersection of $AB$ and $KL$. By symmetry, we can assume that $A$ is between $P$ and $B$.

As usual we denote by $A, B, C$ respectively $a, b, c$ the angles respectively the sides in the triangle $ABC$.

By applying the Menelaus theorem in triangle $ABC$ with transversal $P-L-K$ we obtain

$$\frac{PA}{PB}\frac{KB}{KC}\frac{LC}{LA} = 1 \Leftrightarrow \frac{PA}{c+PA}c\cos(B)b\cos(C)\frac{a\cos(C)}{c\cos(A)} = 1 \Leftrightarrow \frac{PA}{c+PA} = \frac{b\cos(A)}{a\cos(B)}.$$

Therefore

$$PA = \frac{bc\cos(A)}{a\cos(B) - b\ cos(A)}\,.$$

As $ABUV$ is cyclic, we have

$$PU \cdot PV = PA \cdot PB\,.$$

To complete the proof, we need to show that

$$PU \cdot PV = PM \cdot PN\,.$$

We have

$$\begin{aligned}
PU \cdot PV = PM \cdot PN \Leftrightarrow \\
PA \cdot PB = PM \cdot PN \Leftrightarrow \\
PA \cdot (c + PA) = (PA + b\cos(A))(PA + \frac{c}{2}) \Leftrightarrow \\
PA^2 + cPA = PA^2 + bPA\cos(A) + PA\frac{c}{2} + \frac{c}{2}b\cos(A) \Leftrightarrow \\
PA\frac{c}{2} = bPA\cos(A) + \frac{c}{2}b\cos(A) \Leftrightarrow \\
\frac{bc\cos(A)}{a\cos(B) - b\ cos(A)}\frac{c}{2} = b\frac{bc\cos(A)}{a\cos(B) - b\ cos(A)}\cos(A) + \frac{bc\cos(A)}{2} \Leftrightarrow \\
\frac{c}{2(a\cos(B) - b\ cos(A))} = \frac{2b\cos(A)}{2(a\cos(B) - b\ cos(A))} + \frac{a\cos(B) - b\ cos(A)}{2(a\cos(B) - b\ cos(A))} \Leftrightarrow \\
c = 2b\cos(A) + a\cos(B) - b\ cos(A) \Leftrightarrow \\
c = b\cos(A) + a\cos(B)
\end{aligned}$$

which is true.

**OC104**. Given a triangle $ABC$, let $D$ be the midpoint of the side $AC$ and let $M$ be the point on the segment $BD$ so that $BM : MD = 1 : 2$. The rays $AM$ and $CM$ intersect the sides $BC$ and $AB$ at $E$ respectively $F$. We know that $AM \perp CM$. Prove that the quadrangle $AFED$ is cyclic if and only if the median from $A$ in $\Delta ABC$ meets the line $EF$ at a point situated on the circumcircle of $\Delta ABC$.

*Originally question 3 from the 2011 Romania Team Selection Test, Day 4.*

*Solved by D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; and O. Geupel. We give the solution of Oliver Geupel.*

Let $a = BC$, $b = CA$, and $c = AB$.

By Apollonius' theorem, the squared length of the triangle median $BD$ is

$$BD^2 = \frac{2(a^2 + c^2) - b^2}{4}\,.$$

By the assertions $BM : MD = 1 : 2$ and $AM \perp CM$, we have $\frac{2}{3}BD = DM = \frac{b}{2}$. It follows that

$$13b^2 = 8(a^2 + c^2). \tag{1}$$

Using barycentric coordinates, we have

$$\begin{aligned} M &= \frac{1}{6}A + \frac{2}{3}B + \frac{1}{6}C \\ E &= \frac{4}{5}B + \frac{1}{5}C \\ F &= \frac{1}{5}A + \frac{4}{5}B. \end{aligned}$$

Therefore $BE = \frac{a}{5}$, $BF = \frac{c}{5}$, $EF = \frac{b}{5}$, and $EF \parallel AD$.

We show that the quadrilateral $AFED$ is cyclic if and only if

$$b^2 = 2c^2 \quad \text{and} \quad a^2 = \frac{9}{4}c^2. \tag{2}$$

By the law of cosines in $\triangle CDE$ we have

$$\begin{aligned} DE^2 &= CE^2 + CD^2 - 2CE \cdot CD \cos C \\ &= \frac{16}{25}a^2 + \frac{1}{4}b^2 - 2 \cdot \frac{4}{5}a \cdot \frac{1}{2}b \cdot \frac{a^2 + b^2 - c^2}{2ab} \qquad (3) \\ &= \frac{6}{25}a^2 - \frac{3}{20}b^2 + \frac{2}{5}c^2. \qquad (4) \end{aligned}$$

The trapezoid $AFED$ with $EF \neq AD$ is cyclic if and only if it is isosceles, i.e. $DE = \frac{4}{5}c$. By (3), this is equivalent to

$$8a^2 = 5b^2 + 8c^2\,.$$

In view of (1), it simplifies to (2). Consequently, quadrilateral $AFED$ is cyclic if and only if (2).

It suffices to show that the median from $A$ meets the line $EF$ at a point situated on the circumcircle of $\triangle ABC$ if and only if (2) holds.

Let $N$ be the midpoint of $BC$ and let $P$ be the intersection of lines $EF$ and $AN$.

By Apollonius' theorem the squared length of the median $AN$ is

$$AN^2 = \frac{2(b^2+c^2)-a^2}{4}\,.$$

By Menelaus' theorem applied to the line $EF$ in $\triangle ABN$ we have

$$\frac{PN}{PN+AN} = \frac{BF}{AF}\cdot\frac{NE}{BE} = \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8},$$

whence $PN = \frac{3}{5}AN$.

Observe that the point $P$ lies on the circumcircle of triangle $ABC$ if and only if $BN \cdot CN = AN \cdot PN$. This is successively equivalent to

$$\frac{a^2}{4} = \frac{3}{20}(2(b^2+c^2)-a^2)$$

and to $4a^2 = 3(b^2+c^2)$. In view of (1), this simplifies to (2).

This completes the proof.

**OC105**. Let $n > 1$ be an integer, and let $k$ be the number of distinct prime divisors of $n$. Prove that there exists an integer $a, 1 < a < \frac{n}{k}+1$, such that $n \mid a^2 - a$.

*Originally question 2 from 2011 China Team Selection, Quiz 3, Day 1.*

*Solved by Oliver Geupel whose solution is presented below.*

Let $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ be the prime factorization of $n$.

Then, the condition $n \mid a(a-1)$ holds if and only if there are numbers $b_1, b_2, \ldots, b_k \in \{0,1\}$ such that

$$\begin{aligned} a &\equiv b_1 \pmod{p_1^{e_1}} \\ a &\equiv b_2 \pmod{p_2^{e_2}} \\ &\cdots \\ a &\equiv b_k \pmod{p_k^{e_k}}\,. \end{aligned} \tag{5}$$

For $i, j \in \{1, 2, \ldots, k\}$, define

$$b_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases}$$

For $j = 1, \dots, k$, let $\mathcal{S}_j$ denote the following system of simultaneous congruences :

$$\begin{aligned} a &\equiv b_{1j} \pmod{p_1^{e_1}} \\ a &\equiv b_{2j} \pmod{p_2^{e_2}} \\ &\dots \\ a &\equiv b_{kj} \pmod{p_k^{e_k}}. \end{aligned}$$

By the Chinese remainder theorem, there are $k$ numbers $a_1, \dots, a_k$, such that for $j = 1, \dots, k$ it holds $2 \le a_j \le n$, and $a = a_j$ solves the simultaneous congruences $\mathcal{S}_j$.

The $k$ closed intervals

$$\begin{aligned} I_1 &= \left[2,\ \frac{n}{k}\right], \\ I_2 &= \left[\frac{n}{k} + 1,\ 2 \cdot \frac{n}{k}\right], \\ I_3 &= \left[2 \cdot \frac{n}{k} + 1,\ 3 \cdot \frac{n}{k}\right], \\ &\dots, \\ I_k &= \left[(k-1) \cdot \frac{n}{k} + 1,\ n\right] \end{aligned}$$

constitute a disjoint partition of the interval $[2, n]$.

If at least one of the numbers $a_j$, $1 \le j \le k$ belongs to $I_1$, then this number has the required property.

Otherwise, there are two distinct numbers $a', a''$ among $a_1, \dots, a_k$ that belong to the same interval $I_j$ by the Pigeonhole principle.

But then, one of the numbers $|a' - a''|$ and $|a' - a''| + 1$ satisfies a system of simultaneous congruences of the form (5) and belongs to $I_1$.

This completes the proof.