

FOCUS ON...

No. 10

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Some Sequences of Integrals

Introduction

In problem **2520** [2000 : 115], Paul Bracken considered the asymptotic behaviour of some integrals of the form $\int_0^1 (1 + ax + bx^2)^n dx$ as $n \rightarrow \infty$ and a partial solution was given a year later [2001 : 218]. Recently, I proposed two problems on the same topic, derived from results I had obtained when solving this problem (see **3604** [2011 : 46,49 ; 2012 : 32] and **3642** [2011 : 235,237 ; 2012 : 202]). Maybe it is time to complete the work initiated by problem **2520** with a general study of the sequences $\{I_n\}$ where $I_n = \int_0^1 (ax^2 + bx + c)^n dx$ and $ax^2 + bx + c > 0$ for all $x \in [0, 1]$. In what follows, the purpose is to determine a “simple” sequence $\{\omega_n\}$ such that $I_n \sim \omega_n$ as $n \rightarrow \infty$, meaning that $\lim_{n \rightarrow \infty} I_n/\omega_n = 1$. For convenience, we will drop “as $n \rightarrow \infty$ ” after the symbol \sim .

A lemma

First, we give a quick proof of the following result : If $r \in (0, 1]$ and $s \geq r$, then

$$\int_0^r (s^2 - x^2)^n dx \sim \frac{s^{2n+1}}{2} \cdot \sqrt{\frac{\pi}{n}}.$$

The substitution $x = su$ reduces the question to showing that

$$\int_0^\rho (1 - u^2)^n du \sim \frac{1}{2} \cdot \sqrt{\frac{\pi}{n}},$$

where $\rho \in (0, 1]$. Since $\lim_{n \rightarrow \infty} \sqrt{n} \int_\rho^1 (1 - u^2)^n du = 0$ if $\rho < 1$ (the integral being less than $(1 - \rho^2)^n$), all finally amounts to proving that

$$\int_0^1 (1 - u^2)^n du \sim \frac{1}{2} \cdot \sqrt{\frac{\pi}{n}}. \quad (1)$$

With the help of the substitution $u = 1 - 2t$, we calculate

$$\int_0^1 (1 - u^2)^n du = 2^{2n+1} \int_0^{1/2} t^n (1 - t)^n dt = 2^{2n} \int_0^1 t^n (1 - t)^n dt = 2^{2n} \frac{(n!)^2}{(2n + 1)!}$$

and Stirling’s formula $n! \sim n^n e^{-n} \sqrt{2\pi n}$ easily leads to (1).

From now on, we set $\phi(x) = ax^2 + bx + c$ where a, b, c are real numbers ($a \neq 0$) and we assume that $\phi(x) > 0$ for all $x \in [0, 1]$ (so that $c > 0$ and $a + b + c > 0$).

As above, $I_n = \int_0^1 (\phi(x))^n dx$.

The case when ϕ is decreasing on $[0, 1]$

Here are the results in that case :

$$\text{if } b \neq 0, \quad I_n \sim \frac{c^{n+1}}{n|b|} \quad \text{and if } b = 0, \quad I_n \sim \frac{c^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \quad (2)$$

If $b = 0$, then $a < 0$ and $I_n = |a|^n \int_0^1 \left(\frac{c}{|a|} - x^2\right)^n dx$, so the lemma directly gives the announced result.

Now, suppose that $b \neq 0$. Since $\phi(x) = c\left(1 + \frac{b}{c}x + \frac{a}{c}x^2\right)$, it suffices to prove that

$$\lim_{n \rightarrow \infty} n \cdot \int_0^1 (1 + \beta x + \alpha x^2)^n dx = \frac{1}{|\beta|}$$

whenever $\beta \neq 0$ and $x \mapsto \psi(x) = 1 + \beta x + \alpha x^2$ is positive and decreasing on $[0, 1]$ (this implies $\beta < 0$). Let $\varepsilon \in (0, |\beta|)$. Since $\lim_{x \rightarrow 0^+} \frac{1 - \psi(x)}{x} = |\beta|$, we can choose $\delta \in (0, 1)$, small enough to ensure that for $x \in [0, \delta]$,

$$0 < 1 - (|\beta| + \varepsilon)x \leq \psi(x) \leq 1 - (|\beta| - \varepsilon)x. \quad (3)$$

Because $0 \leq n \cdot \int_\delta^1 (\psi(x))^n dx \leq n(\psi(\delta))^n$ and $0 < \psi(\delta) < 1$, we have

$$\lim_{n \rightarrow \infty} n \cdot \int_\delta^1 (\psi(x))^n dx = 0.$$

Let $J_n = \int_0^1 (\psi(x))^n dx$ and $K_n = \int_0^\delta (\psi(x))^n dx$. From (3), we obtain

$$\frac{n}{n+1} \cdot \frac{1 - \rho_1^{n+1}}{|\beta| + \varepsilon} \leq nK_n \leq \frac{n}{n+1} \cdot \frac{1 - \rho_2^{n+1}}{|\beta| - \varepsilon}$$

where $\rho_1 = 1 - (|\beta| + \varepsilon)\delta$, $\rho_2 = 1 - (|\beta| - \varepsilon)\delta$ are in $(0, 1)$.

Observing that $nJ_n = nK_n + n \int_\delta^1 (\psi(x))^n dx$, we readily deduce that

$$\limsup_{n \rightarrow \infty} nJ_n \leq \frac{1}{|\beta| - \varepsilon} \quad \text{and} \quad \liminf_{n \rightarrow \infty} nJ_n \geq \frac{1}{|\beta| + \varepsilon}$$

and since this holds for all $\varepsilon \in (0, |\beta|)$, $\frac{1}{|\beta|} \leq \liminf_{n \rightarrow \infty} nJ_n \leq \limsup_{n \rightarrow \infty} nJ_n \leq \frac{1}{|\beta|}$.

The result follows.

The other cases

(a) If ϕ is increasing on $[0, 1]$, the change of variables $x = 1 - y$ shows that

$$I_n = \int_0^1 (a + b + c - (2a + b)y + ay^2)^n dy$$

and applying (2) gives

$$\text{if } 2a + b \neq 0, \quad I_n \sim \frac{(a + b + c)^{n+1}}{n(2a + b)}, \quad (4)$$

$$\text{if } 2a + b = 0, \quad I_n \sim \frac{(c - a)^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \quad (5)$$

(b) If ϕ attains its minimum on $(0, 1)$, then $a > 0$ and $0 < \frac{b}{2a} < 1$ so that $b < 0$.

Also note that $\Delta = b^2 - 4ac < 0$. Let $\mu = \frac{\sqrt{|\Delta|}}{2a}$. Then,

$$\begin{aligned} I_n &= a^n \int_0^1 \left(\left(x + \frac{b}{2a} \right)^2 + \mu^2 \right)^n dx = a^n \int_{\frac{b}{2a}}^{1+\frac{b}{2a}} (y^2 + \mu^2)^n dy \\ &= a^n \left(\int_0^{1+\frac{b}{2a}} (y^2 + \mu^2)^n dy + \int_0^{-\frac{b}{2a}} (y^2 + \mu^2)^n dy \right). \end{aligned}$$

Now, from (4, 5) we obtain

$$\int_0^k (y^2 + \mu^2)^n dy = k \int_0^1 (k^2 x^2 + \mu^2)^n dx \sim \frac{(\mu^2 + k^2)^{n+1}}{2kn}$$

for positive k . It is then straightforward to deduce

$$\text{if } |b| < a: \quad I_n \sim \frac{(a + b + c)^{n+1}}{n(2a + b)},$$

$$\text{if } |b| > a: \quad I_n \sim \frac{c^{n+1}}{n|b|},$$

$$\text{if } |b| = a: \quad I_n \sim \frac{2c^{n+1}}{n|b|}.$$

(c) There remains the case when ϕ attains its maximum on $(0, 1)$. Then

$$a < 0, \quad 0 < \frac{-b}{2a} < 1 \quad \text{and} \quad \Delta = b^2 - 4ac > 0.$$

Completing the square as in the previous case, we arrive at

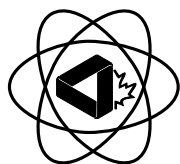
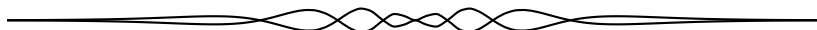
$$I_n = |a|^n \left(\int_0^{1+\frac{b}{2a}} (\nu^2 - y^2)^n dy + \int_0^{-\frac{b}{2a}} (\nu^2 - y^2)^n dy \right),$$

where $\nu = \frac{\sqrt{\Delta}}{2|a|}$. Using the lemma, we obtain

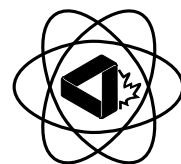
$$I_n \sim \sqrt{\frac{\pi}{n|a|}} \left(c - \frac{b^2}{4a} \right)^{n+\frac{1}{2}}.$$

Exercise

The obvious one is to find an alternative solution to **3604** and **3642** with the help of the results established above.



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