FOCUS ON...

No. 10
Michel Bataille
Some Sequences of Integrals

Introduction
In problem 2520 [2000 : 115], Paul Bracken considered the asymptotic behaviour of some integrals of the form \( \int_0^1 (1 + ax + bx^2)^n \, dx \) as \( n \to \infty \) and a partial solution was given a year later [2001 : 218]. Recently, I proposed two problems on the same topic, derived from results I had obtained when solving this problem (see 3604 [2011 : 46,49 ; 2012 : 32] and 3642 [2011 : 235,237 ; 2012 : 202]). Maybe it is time to complete the work initiated by problem 2520 with a general study of the sequences \( \{I_n\} \) where \( I_n = \int_0^1 (ax^2 + bx + c)^n \, dx \) and \( ax^2 + bx + c > 0 \) for all \( x \in [0,1] \). In what follows, the purpose is to determine a “simple” sequence \( \{\omega_n\} \) such that \( I_n \sim \omega_n \) as \( n \to \infty \), meaning that \( \lim_{n \to \infty} I_n/\omega_n = 1 \). For convenience, we will drop “as \( n \to \infty \)” after the symbol \( \sim \).

A lemma
First, we give a quick proof of the following result : If \( r \in (0,1) \) and \( s \geq r \), then
\[
\int_0^r (s^2 - x^2)^n \, dx \sim \frac{s^{2n+1}}{2} \cdot \sqrt{\frac{\pi}{n}}.
\]
The substitution \( x = su \) reduces the question to showing that
\[
\int_0^\rho (1 - u^2)^n \, du \sim \frac{1}{2} \cdot \sqrt{\frac{\pi}{n}},
\]
where \( \rho \in (0,1) \). Since \( \lim_{n \to \infty} \sqrt{n} \int_0^1 (1 - u^2)^n \, du = 0 \) if \( \rho < 1 \) (the integral being less than \( (1 - \rho^2)^n \)), all finally amounts to proving that
\[
\int_0^1 (1 - u^2)^n \, du \sim \frac{1}{2} \cdot \sqrt{\frac{\pi}{n}}.
\]
(1)
With the help of the substitution \( u = 1 - 2t \), we calculate
\[
\int_0^1 (1 - u^2)^n \, du = 2^{2n+1} \int_0^{1/2} t^n (1 - t)^n \, dt = 2^{2n} \int_0^1 t^n (1 - t)^n \, dt = 2^{2n} \frac{(n!)^2}{(2n + 1)!}
\]
and Stirling’s formula \( n! \sim n^n e^{-n} \sqrt{2\pi n} \) easily leads to (1).

Copyright © Canadian Mathematical Society, 2015
From now on, we set \( \phi(x) = ax^2 + bx + c \) where \( a, b, c \) are real numbers \( (a \neq 0) \) and we assume that \( \phi(x) > 0 \) for all \( x \in [0,1] \) (so that \( c > 0 \) and \( a + b + c > 0 \)).

As above, \( I_n = \int_0^1 (\phi(x))^n \, dx \).

**The case when \( \phi \) is decreasing on \([0,1]\)**

Here are the results in that case:

if \( b \neq 0 \), \( I_n \sim \frac{c^{n+1}}{n} \) \( |\beta| \) and if \( b = 0 \), \( I_n \sim \frac{c^{n+\frac{1}{2}} \cdot \sqrt{\pi}}{2n|a|} \). \( (2) \)

If \( b = 0 \), then \( a < 0 \) and \( I_n = \int_0^1 (1 + \frac{\beta}{c}x + \frac{\alpha}{c}x^2)^n \, dx \), so the lemma directly gives the announced result.

Now, suppose that \( b \neq 0 \). Since \( \phi(x) = c \left( 1 + \frac{\beta}{c}x + \frac{\alpha}{c}x^2 \right) \), it suffices to prove that

\[
\lim_{n \to \infty} \frac{n}{\beta} \int_0^1 (1 + \frac{\beta}{c}x + \frac{\alpha}{c}x^2)^n \, dx = \frac{1}{|\beta|}
\]

whenever \( \beta \neq 0 \) and \( x \to \psi(x) = 1 + \frac{\beta}{c}x + \frac{\alpha}{c}x^2 \) is positive and decreasing on \([0,1]\) (this implies \( \beta < 0 \)). Let \( \varepsilon \in (0,|\beta|) \). Since \( \lim_{x \to 0^+} \frac{1-\psi(x)}{x} = |\beta| \), we can choose \( \delta \in (0,1) \), small enough to ensure that for \( x \in [0,\delta] \),

\[
0 < 1 - (|\beta| + \varepsilon)x \leq \psi(x) \leq 1 - (|\beta| - \varepsilon)x.
\] \( (3) \)

Because \( 0 \leq n \cdot \int_0^1 (\psi(x))^n \, dx \leq n(\psi(\delta))^n \) and \( 0 < \psi(\delta) < 1 \), we have

\[
\lim_{n \to \infty} n \cdot \int_0^1 (\psi(x))^n \, dx = 0.
\]

Let \( J_n = \int_0^1 (\psi(x))^n \, dx \) and \( K_n = \int_\delta^1 (\psi(x))^n \, dx \). From (3), we obtain

\[
\frac{n}{n+1} \cdot \frac{1-\rho_1^{n+1}}{|\beta| + \varepsilon} \leq nK_n \leq \frac{n}{n+1} \cdot \frac{1-\rho_2^{n+1}}{|\beta| - \varepsilon}
\]

where \( \rho_1 = 1 - (|\beta| + \varepsilon)\delta \), \( \rho_2 = 1 - (|\beta| - \varepsilon)\delta \) are in \((0,1)\).

Observing that \( nJ_n = nK_n + n \int_\delta^1 (\psi(x))^n \, dx \), we readily deduce that

\[
\limsup_{n \to \infty} nJ_n \leq \frac{1}{|\beta| - \varepsilon} \quad \text{and} \quad \liminf_{n \to \infty} nJ_n \geq \frac{1}{|\beta| + \varepsilon}
\]

and since this holds for all \( \varepsilon \in (0,|\beta|) \), \( \frac{1}{|\beta|} \leq \liminf_{n \to \infty} nJ_n \leq \limsup_{n \to \infty} nJ_n \leq \frac{1}{|\beta|} \).

The result follows.

*Crux Mathematicorum, Vol. 40(1), January 2014*
The other cases

(a) If $\phi$ is increasing on $[0,1]$, the change of variables $x = 1 - y$ shows that

$$I_n = \int_0^1 (a + b + c - (2a + b)y + ay^2)^n dy$$

and applying (2) gives

$$\begin{align*}
\text{if } 2a + b \neq 0, & \quad I_n \sim \frac{(a + b + c)^{n+1}}{n(2a + b)}, \\
\text{if } 2a + b = 0, & \quad I_n \sim \frac{(c-a)^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}.
\end{align*}$$

(b) If $\phi$ attains its minimum on $(0,1)$, then $a > 0$ and $0 < -\frac{b}{2a} < 1$ so that $b < 0$. Also note that $\Delta = b^2 - 4ac < 0$. Let $\mu = \sqrt{|\Delta|}$. Then,

$$I_n = a^n \int_0^1 \left( \left( x + \frac{b}{2a} \right)^2 + \mu^2 \right)^n dx = a^n \int_{\frac{b}{2a}}^{1+\frac{b}{2a}} (y^2 + \mu^2)^n dy$$

$$= a^n \left( \int_{0}^{1+\frac{b}{2a}} (y^2 + \mu^2)^n dy + \int_{-\frac{b}{2a}}^{\frac{b}{2a}} (y^2 + \mu^2)^n dy \right).$$

Now, from (4, 5) we obtain

$$\int_0^k (y^2 + \mu^2)^n dy = k \int_0^1 (k^2x^2 + \mu^2)^n dx \sim \frac{(\mu^2 + k^2)^{n+1}}{2kn}$$

for positive $k$. It is then straightforward to deduce

$$\begin{align*}
\text{if } |b| < a : & \quad I_n \sim \frac{(a + b + c)^{n+1}}{n(2a + b)}, \\
\text{if } |b| > a : & \quad I_n \sim \frac{c^{n+1}}{n|b|}, \\
\text{if } |b| = a : & \quad I_n \sim \frac{2c^{n+1}}{n|b|}.
\end{align*}$$

(c) There remains the case when $\phi$ attains its maximum on $(0,1)$. Then

$$a < 0, \quad 0 < -\frac{b}{2a} < 1 \quad \text{and} \quad \Delta = b^2 - 4ac > 0.$$
where \( \nu = \frac{\sqrt{3}}{2n}\). Using the lemma, we obtain

\[
I_n \sim \sqrt{\frac{n}{n|a|}} \left( c - \frac{b^2}{4a} \right)^{n+\frac{1}{2}}.
\]

Exercise

The obvious one is to find an alternative solution to 3604 and 3642 with the help of the results established above.

---

ATOM Volume IX : The CAUT Problems
by Edward Barbeau (University of Toronto)

This book contains over sixty problems that were originally published in the CAUT Bulletin, published by the Canadian Association of University Teachers. The Canadian Mathematical Society is grateful to the Canadian Association of University Teachers for granting permission for these problems to be included in this ATOM volume. The reader might ask how mathematical problems wound up in a publication whose articles generally deal with matters of university governance, academic policies and conditions of employment of the faculty and librarians of Canada’s colleges and universities.

Most of these problems are not original; the ideas come from a number of sources, some of them school texts. New problems appear on the scene regularly, and a couple of them are included here. I hope that readers enjoy trying their hand at them.

There are currently 13 booklets in the series. For information on titles in this series and how to order, visit the ATOM page on the CMS website: