FOCUS ON...

No. 10

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Some Sequences of Integrals

Introduction

In problem **2520** [2000 : 115], Paul Bracken considered the asymptotic behaviour of some integrals of the form $\int_0^1 (1+ax+bx^2)^n\,dx$ as $n\to\infty$ and a partial solution was given a year later [2001 : 218]. Recently, I proposed two problems on the same topic, derived from results I had obtained when solving this problem (see **3604** [2011 : 46,49 ; 2012 : 32] and **3642** [2011 : 235,237 ; 2012 : 202]). Maybe it is time to complete the work initiated by problem **2520** with a general study of the sequences $\{I_n\}$ where $I_n = \int_0^1 (ax^2+bx+c)^n\,dx$ and $ax^2+bx+c>0$ for all $x\in[0,1]$. In what follows, the purpose is to determine a "simple" sequence $\{\omega_n\}$ such that $I_n \sim \omega_n$ as $n\to\infty$, meaning that $\lim_{n\to\infty} I_n/\omega_n = 1$. For convenience, we will drop "as $n\to\infty$" after the symbol $\sim$.

A lemma

First, we give a quick proof of the following result : If $r\in(0,1]$ and $s\ge r$, then

$$\int_0^r (s^2-x^2)^n\,dx \ \sim\ \frac{s^{2n+1}}{2}\cdot\sqrt{\frac{\pi}{n}}.$$

The substitution $x=su$ reduces the question to showing that

$$\int_0^\rho (1-u^2)^n\,du \ \sim\ \frac{1}{2}\cdot\sqrt{\frac{\pi}{n}},$$

where $\rho\in(0,1]$. Since $\lim_{n\to\infty}\sqrt{n}\int_\rho^1 (1-u^2)^n\,du = 0$ if $\rho<1$ (the integral being less than $(1-\rho^2)^n$), all finally amounts to proving that

$$\int_0^1 (1-u^2)^n\,du \ \sim\ \frac{1}{2}\cdot\sqrt{\frac{\pi}{n}}. \tag{1}$$

With the help of the substitution $u=1-2t$, we calculate

$$\int_0^1 (1-u^2)^n\,du = 2^{2n+1}\int_0^{1/2} t^n(1-t)^n\,dt = 2^{2n}\int_0^1 t^n(1-t)^n\,dt = 2^{2n}\frac{(n!)^2}{(2n+1)!}$$

and Stirling's formula $n!\ \sim\ n^ne^{-n}\sqrt{2\pi n}$ easily leads to (1).

From now on, we set $\phi(x) = ax^2 + bx + c$ where a, b, c are real numbers ($a \neq 0$) and we assume that $\phi(x) > 0$ for all $x \in [0, 1]$ (so that $c > 0$ and $a + b + c > 0$). As above, $I_n = \displaystyle\int_0^1 (\phi(x))^n \, dx$.

The case when ϕ is decreasing on $[0, 1]$

Here are the results in that case :

$$\text{if } b \neq 0, \quad I_n \sim \frac{c^{n+1}}{n|b|} \qquad \text{and if } b = 0, \quad I_n \sim \frac{c^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \tag{2}$$

If $b = 0$, then $a < 0$ and $I_n = |a|^n \displaystyle\int_0^1 \left(\frac{c}{|a|} - x^2\right)^n dx$, so the lemma directly gives the announced result.

Now, suppose that $b \neq 0$. Since $\phi(x) = c\left(1 + \frac{b}{c}x + \frac{a}{c}x^2\right)$, it suffices to prove that

$$\lim_{n\to\infty} n \cdot \int_0^1 (1 + \beta x + \alpha x^2)^n = \frac{1}{|\beta|}$$

whenever $\beta \neq 0$ and $x \mapsto \psi(x) = 1 + \beta x + \alpha x^2$ is positive and decreasing on $[0, 1]$ (this implies $\beta < 0$). Let $\varepsilon \in (0, |\beta|)$. Since $\lim_{x\to 0^+} \frac{1-\psi(x)}{x} = |\beta|$, we can choose $\delta \in (0, 1)$, small enough to ensure that for $x \in [0, \delta]$,

$$0 < 1 - (|\beta| + \varepsilon)x \leq \psi(x) \leq 1 - (|\beta| - \varepsilon)x. \tag{3}$$

Because $0 \leq n \cdot \displaystyle\int_\delta^1 (\psi(x))^n \, dx \leq n(\psi(\delta))^n$ and $0 < \psi(\delta) < 1$, we have

$$\lim_{n\to\infty} n \cdot \int_\delta^1 (\psi(x))^n \, dx = 0.$$

Let $J_n = \displaystyle\int_0^1 (\psi(x))^n \, dx$ and $K_n = \displaystyle\int_0^\delta (\psi(x))^n \, dx$. From (3), we obtain

$$\frac{n}{n+1} \cdot \frac{1 - \rho_1^{n+1}}{|\beta| + \varepsilon} \leq nK_n \leq \frac{n}{n+1} \cdot \frac{1 - \rho_2^{n+1}}{|\beta| - \varepsilon}$$

where $\rho_1 = 1 - (|\beta| + \varepsilon)\delta, \quad \rho_2 = 1 - (|\beta| - \varepsilon)\delta$ are in $(0, 1)$.

Observing that $nJ_n = nK_n + n\displaystyle\int_\delta^1 (\psi(x))^n \, dx$, we readily deduce that

$$\limsup_{n\to\infty} \; nJ_n \leq \frac{1}{|\beta| - \varepsilon} \quad \text{and} \quad \liminf_{n\to\infty} \; nJ_n \geq \frac{1}{|\beta| + \varepsilon}$$

and since this holds for all $\varepsilon \in (0, |\beta|)$, $\frac{1}{|\beta|} \leq \liminf\limits_{n\to\infty} \; nJ_n \leq \limsup\limits_{n\to\infty} \; nJ_n \leq \frac{1}{|\beta|}$.

The result follows.

The other cases

(a) If ϕ is increasing on $[0,1]$, the change of variables $x = 1 - y$ shows that

$$I_n = \int_0^1 (a + b + c - (2a + b)y + ay^2)^n \, dy$$

and applying (2) gives

$$\text{if } 2a + b \neq 0, \quad I_n \sim \frac{(a+b+c)^{n+1}}{n(2a+b)}, \tag{4}$$

$$\text{if } 2a + b = 0, \quad I_n \sim \frac{(c-a)^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \tag{5}$$

(b) If ϕ attains its minimum on $(0,1)$, then $a > 0$ and $0 < \frac{-b}{2a} < 1$ so that $b < 0$. Also note that $\Delta = b^2 - 4ac < 0$. Let $\mu = \frac{\sqrt{|\Delta|}}{2a}$. Then,

$$I_n = a^n \int_0^1 \left(\left(x + \frac{b}{2a} \right)^2 + \mu^2 \right)^n dx = a^n \int_{\frac{b}{2a}}^{1+\frac{b}{2a}} (y^2 + \mu^2)^n \, dy$$

$$= a^n \left(\int_0^{1+\frac{b}{2a}} (y^2 + \mu^2)^n \, dy + \int_0^{\frac{-b}{2a}} (y^2 + \mu^2)^n \, dy \right).$$

Now, from $(4, 5)$ we obtain

$$\int_0^k (y^2 + \mu^2)^n \, dy = k \int_0^1 \left(k^2 x^2 + \mu^2 \right)^n \, dx \sim \frac{(\mu^2 + k^2)^{n+1}}{2kn}$$

for positive k. It is then straightforward to deduce

$$\text{if } |b| < a: \quad I_n \sim \frac{(a+b+c)^{n+1}}{n(2a+b)},$$

$$\text{if } |b| > a: \quad I_n \sim \frac{c^{n+1}}{n|b|},$$

$$\text{if } |b| = a: \quad I_n \sim \frac{2c^{n+1}}{n|b|}.$$

(c) There remains the case when ϕ attains its maximum on $(0,1)$. Then

$$a < 0, \quad 0 < \frac{-b}{2a} < 1 \quad \text{and} \quad \Delta = b^2 - 4ac > 0.$$

Completing the square as in the previous case, we arrive at

$$I_n = |a|^n \left(\int_0^{1+\frac{b}{2a}} (\nu^2 - y^2)^n \, dy + \int_0^{\frac{-b}{2a}} (\nu^2 - y^2)^n \, dy \right),$$

where $\nu = \frac{\sqrt{\Delta}}{2|a|}$. Using the lemma, we obtain

$$I_n \sim \sqrt{\frac{\pi}{n|a|}} \left(c - \frac{b^2}{4a}\right)^{n+\frac{1}{2}}.$$

Exercise

The obvious one is to find an alternative solution to **3604** and **3642** with the help of the results established above.