CC102. Dans le pentagone $ABCDE$, les angles B et D sont droits. Démontrer que le périmètre du triangle ACE est supérieur ou égal à $2BD$.

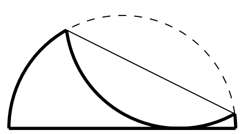
CC103. Soit deux nombres rationnels a et b tels que $\sqrt{a}+\sqrt{b}+\sqrt{ab}$ est aussi rationnel. Montrez que $\sqrt{a}$ and $\sqrt{b}$ sont aussi des rationnels.

CC104. Comparer l'aire du cercle inscrit dans un carré à l'aire du cercle circonscrit au carré.

CC105. En partant du fait que $3.3025 < \log_{10} 2007 < 3.3026$, déterminer le premier chiffre à gauche dans l'écriture décimale de 2007^{1000}.

CONTEST CORNER SOLUTIONS

CC51. A semicircular piece of paper with radius 2 is creased and folded along a chord so that the arc is tangent to the diameter as shown in the diagram. If the contact point of the arc divides the diameter in the ratio 3 : 1, determine the length of the crease.



Originally problem 5 of 1997 Invitational Mathematics Challenge, Grade 11.

Solved by L. Bobo; R. Girard; R. Hess; S. Muralidharan; A. Plaza; N. Stanciu; and T. Zvonaru. We present the solution by S. Muralidharan.

Let $O(0,0)$ be the centre of the semi-circle with radius 2. Let C be the centre of the circle formed by extending the folded portion. Let R be the point of tangency. Let DE be the length of the crease. Let S be the point where the line joining the centres intersects the crease.

The folded circular portion also has radius 2 and since it touches the diameter at the point $(1,0)$, its centre is at $C(1,2)$. Since these are equal circles, the line joining their centres and the common chord (the crease) bisect each other. Thus, $OS = \frac{OC}{2} = \frac{\sqrt{5}}{2}$ and hence

$$DE = 2DS = 2\sqrt{OD^2 - OS^2} = 2\sqrt{4 - \frac{5}{4}} = \sqrt{11}.$$

CC52. There are some marbles in a bowl. Alphonse, Beryl and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can Beryl and Colleen work together and force Alphonse to lose ?

Originally 2002 Canadian Open Mathematics Challenge, problem B3b.

Solved by R. Hess ; and S. Muralidharan. We present the solution by Richard Hess.

We claim that Beryl and Colleen can force Alphonse to lose for all N except $N = 2, 3, 4, 7$, or 8.

At $N = 2$, Alphonse leaves 1.

At $N = 3$ or 4, Alphonse leaves 2.

At $N = 7$ or 8, Alphonse leaves 6 after which Beryl and Colleen must leave 2, 3 or 4.

For $N = 5$ or 6, regardless of what Alphonse takes, Beryl and Colleen can work it so that when Alphonse's turn arrives there is only one marble left.

For $N = 9$ or 10, Alphonse must leave 7, 8 or 9 from which Beryl and Colleen can force 5 or 6.

For $N = 4k$ where $k > 2$, Alphonse must leave either $4k - 1$ or $4k - 2$ from which Beryl and Colleen can force $4(k - 1) + 1$ or $4(k - 2) + 2$.

For $N = 4k + 1$, Alphonse must leave either $4k$ or $4k - 1$ from which Beryl and Colleen can force $4(k - 1) + 2$ or $4(k - 1) + 1$.

For $N = 4k + 2$, Alphonse must leave either $4k + 1$ or $4k$ from which Beryl and Colleen can force $4(k - 1) + 2$ or $4(k - 1) + 1$.

For $N = 4k + 3$, Alphonse must leave either $4k + 2$ or $4k + 1$ from which Beryl and Colleen can force $4(k - 1) + 2$.

In all cases for $N \geq 11$, Alphonse will always be faced with a new value of the form $4t + 1$ or $4t + 2$ on his next turn eventually forcing him to $N = 5$ or 6 and a loss.

CC53. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

Originally problem 8 of 2009 Sun Life Financial Repêchage Competition.

Solved by R. Hess ; and T. Zvonaru. We present the solution by Titu Zvonaru.

Starting with the identity, $3^{n+1} = 3^n + 3^n + 3^n$, we want to find a positive integer, n, so that $7 \mid (n + 1)$, $6 \mid n$, $5 \mid n$, and $4 \mid n$. The latter three relations combine to $60 \mid n$, and we can quickly find that $n = 300$ works. Thus we can write $(3^{75})^4 +$

$(3^{60})^5 + (3^{50})^6 = (3^{43})^7$. Multiplying by m^{420} for any positive integer, m, gives us an infinity of quadruples

$$(a, b, c, d) = (m^{105}3^{75}, m^{84}3^{60}, m^{70}3^{50}, m^{60}3^{43})$$

of positive integers, each of which is a solution to the equation from the statement.

CC54. Let k, l, m, n be positive integers such that $k + l + m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

The summation in the left-hand side runs over all ordered partitions of n into three integers p, q, r such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

Originally 2004 Memorial University Undergraduate Mathematics Competition, Question 3.

Solved by J. L. Díaz-Barrero; S. Muralidharan; A. Plaza; and D. Văcaru. We present the solution by S. Muralidharan below.

We will prove the result by counting the number of ways of forming a team of n people from a group containing k Canadians, l Americans and m Australians.

The number of teams in which there are p Canadians, q Americans and r Australians where $0 \leq p \leq k$, $0 \leq q \leq l$ and $0 \leq r \leq m$ and $p + q + r = n$ is $\binom{k}{p}\binom{l}{q}\binom{m}{r}$. Thus the total number of teams is $\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r}$.

On the other hand a team of n people from the group containing $k+l+m$ people can be formed in $\binom{k+l+m}{n}$ ways. It follows that,

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

CC55. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

Originally 2011 APICS Math Competition, Question 3.

Solved by Š. Arslanagić; M. Coiculescu; J. L. Díaz-Barrero; R. Hess; D. E. Manes; S. Muralidharan; P. Perfetti; G. Tsapakidis; D. Văcaru; and T. Zvonaru. We present two solutions.

Solution 1 by José Luis Díaz-Barrero.

Let $S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$. Then

$$\begin{aligned} S &= 2\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma}\right) + \left(\frac{\alpha-1}{1-\alpha} + \frac{\beta-1}{1-\beta} + \frac{\gamma-1}{1-\gamma}\right) \\ &= 2\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma}\right) - 3. \end{aligned}$$

Since α, β, γ are the roots of $P(x) = x^3 - x - 1$, then $1-\alpha, 1-\beta, 1-\gamma$ are the roots of $P(1-x) = -x^3 + 3x^2 - 2x - 1$ or the roots of $Q(x) = x^3 - 3x^2 + 2x + 1$. Hence, $\frac{1}{1-\alpha}$, $\frac{1}{1-\beta}$ and $\frac{1}{1-\gamma}$ are the roots of $Q(\frac{1}{x}) = 0$ or $x^3 + 2x^2 - 3x + 1$. By Viète formulae, we then have

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = -2,$$

so $S = 2(-2) - 3 = -7$.

Solution 2 composed of solutions by S. Muralidharan and Titu Zvonaru.

We set $y = \dfrac{1+x}{1-x}$. Then $x = \dfrac{y-1}{y+1}$ and then we have

$$\begin{aligned} \left(\frac{y-1}{y+1}\right)^3 - \frac{y-1}{y+1} - 1 &= 0, \\ (y-1)^3 - (y-1)(y+1)^2 - (y+1)^3 &= 0, \\ -6y^2 - 2 - (y^2-1)(y+1) &= 0, \\ y^3 + 7y^2 - y + 1 &= 0. \end{aligned}$$

The last equation has roots $y_1 = \frac{1+\alpha}{1-\alpha}, y_2 = \frac{1+\beta}{1-\beta}$ and $y_3 = \frac{1+\gamma}{1-\gamma}$. Hence, by Viète's formula, we have

$$y_1 + y_2 + y_3 = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = -7.$$